

Kovalevskaya exponents and the space of initial conditions of a quasi-homogeneous vector field

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July 6, 2014; Last modified Aug 15 2015

Abstract

Formal series solutions and the Kovalevskaya exponents of a quasi-homogeneous polynomial system of differential equations are studied by means of a weighted projective space and dynamical systems theory. A necessary and sufficient condition for the series solution to be a convergent Laurent series is given, which improve the well known Painlevé test. In particular, if a given system has the Painlevé property, an algorithm to construct Okamoto's space of initial conditions is given. The space of initial conditions is obtained by weighted blow-ups of the weighted projective space, where the weights for the blow-ups are determined by the Kovalevskaya exponents. The results are applied to the first Painlevé hierarchy ($2m$ -th order first Painlevé equation).

Keywords: quasi-homogeneous vector field; weighted projective space; Kovalevskaya exponent; the first Painlevé hierarchy

1 Introduction

A system of polynomial differential equations

$$\frac{dx_i}{dz} = f_i(x_1, \dots, x_m, z) + g_i(x_1, \dots, x_m, z), \quad i = 1, \dots, m, \quad (1.1)$$

is considered, where $(x_1, \dots, x_m, z) \in \mathbb{C}^{m+1}$, and f_i and g_i satisfy certain conditions on the quasi-homogeneity (see assumptions (A1) to (A3) in Sec.2.1), for which Kovalevskaya exponents are well defined. The first, second, fourth Painlevé equations and the first Painlevé hierarchy satisfy these conditions. This system is investigated with the aid of the $m+1$ dimensional weighted projective space $\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s)$ with the positive weight $(p_1, \dots, p_m, r, s) \in \mathbb{Z}_{>0}^{m+2}$ determined by the quasi-homogeneity of the system. The space $\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s)$ is decomposed as

$$\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s) = \mathbb{C}^{m+1}/\mathbb{Z}_s \cup \mathbb{C}P^m(p_1, \dots, p_m, r), \quad (\text{disjoint}).$$

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This implies that the space is a compactification of $\mathbb{C}^{m+1}/\mathbb{Z}_s$ obtained by attaching the m -dim weighted projective space $\mathbb{C}P^m(p_1, \dots, p_m, r)$ at infinity. The lift $\mathbb{C}^{m+1} = \{(x_1, \dots, x_m, z)\}$ of the quotient $\mathbb{C}^{m+1}/\mathbb{Z}_s$ is a natural phase space, on which the system (1.1) is given. The system is also well defined on the quotient $\mathbb{C}^{m+1}/\mathbb{Z}_s$ because it is invariant under the \mathbb{Z}_s action due to the quasi-homogeneity assumptions. Then, the system is continuously extended to the codimension one space $\mathbb{C}P^m(p_1, \dots, p_m, r)$ attached at infinity. The asymptotic behavior of solutions of the system will be captured by investigating behavior around the “infinity set” $\mathbb{C}P^m(p_1, \dots, p_m, r)$; The space $\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s)$ gives a suitable compactification of the phase space of the system.

A formal series solution of the form

$$x_i(z) = c_i(z - z_0)^{-p_i} + a_{i,1}(z - z_0)^{-p_i+1} + a_{i,2}(z - z_0)^{-p_i+2} + \dots \quad (1.2)$$

will be considered, where z_0 is an arbitrary constant (movable singularity), c_i is a constant and $a_{i,n}$ may include $\log(z - z_0)$. If $a_{i,n}$ is independent of z and the series is convergent, it provides a Laurent series solution. If coefficients c_i and $a_{i,n}$ include n arbitrary parameters other than z_0 , it represents an $n + 1$ -parameter family of solutions. It will be shown that there exists a singularity (in the sense of a foliation defined by integral curves) of the system on the “infinity set” $\mathbb{C}P^{m+1}(p_1, \dots, p_m, r)$, to which the family (1.2) of formal series solutions approaches as $z \rightarrow z_0$ (Lemma 3.3). Hence, the asymptotics of (1.2) as $z \rightarrow z_0$ can be investigated by local analysis around the singularity. In particular, the normal form theory of dynamical systems will play an important role. It will be proved in Thm.3.4 that the eigenvalues of the Jacobi matrix at the singularity coincide with the Kovalevskaya exponents, which implies that the Kovalevskaya exponents are invariant under smooth coordinate transformations. By combining the weighted projective space $\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s)$, the Kovalevskaya exponents and the normal form theory, a necessary and sufficient condition for the series solutions (1.2) to be a convergent Laurent series will be given, which refines the classical Painlevé test [1, 14]. To give the necessary and sufficient condition, it will be shown that the system (1.1) has formal solutions of the form

$$x_i(z) = c_i T^{-p_i} (1 + \tilde{h}_i(\alpha_2 T^{\lambda_2}, \dots, \alpha_m T^{\lambda_m}, z_0 T^r, \varepsilon_0 T^s)), \quad T := z - z_0, \quad (1.3)$$

where \tilde{h}_i is a formal power series in the arguments, whose coefficients are polynomial in $\log T$, $\alpha_2, \dots, \alpha_m, z_0, \varepsilon_0$ are arbitrary parameters and $\lambda_2, \dots, \lambda_m$ are Kovalevskaya exponents other than the trivial exponent $\lambda_1 = -1$ (Lemma 3.6). Suppose that $\text{Re}(\lambda_i) \leq 0$ for $i = 2, \dots, k$ and $\text{Re}(\lambda_i) > 0$ for $i = k + 1, \dots, m$. The unstable manifold theorem proves that

$$x_i(z) = c_i T^{-p_i} (1 + \tilde{h}_i(0, \dots, 0, \alpha_{k+1} T^{\lambda_{k+1}}, \dots, \alpha_m T^{\lambda_m}, z_0 T^r, \varepsilon_0 T^s)), \quad (1.4)$$

is a convergent series. Further, the normal form theory provides a necessary and sufficient condition for it to be the Laurent series without $\log T$ (Prop.3.5). One of the necessary condition is that all Kovalevskaya exponents λ_i with $\text{Re}(\lambda_i) > 0$ are positive integers, as is well known as the Painlevé test.

As mentioned, the family of series solutions (1.2) tends to the singularity on the “infinity set” as $z \rightarrow z_0$. If the series is a convergent Laurent series, an algorithm to resolve the singularity by a weighted blow-up will be given. The weight for the weighted blow-up is determined by the Kovalevskaya exponents. In particular, if a given system has the Painlevé property in the sense that any solutions are meromorphic, our method provides an algorithm to construct the space of initial conditions. For a polynomial system, a manifold $\mathcal{M}(z)$ is called the space of initial conditions if any solutions of the system give global holomorphic sections of the fiber bundle $\mathcal{P} = \{(x, z) \mid x \in \mathcal{M}(z), z \in \mathbb{C}\}$ over \mathbb{C} . If the system has n -types of Laurent series solutions, then the space of initial conditions is obtained by n -times weighted blow-up, which proves that $\mathcal{M}(z)$ is a smooth algebraic variety obtained by gluing the spaces of the form $\mathbb{C}^m / \mathbb{Z}_{p_j}$ with some integers p_j .

In our previous papers [3, 4], weighted projective spaces and dynamical systems theory are applied to the study of the 2-dim Painlevé equations (the first to sixth Painlevé equations). In particular, it is shown that these equations are linearized by a local analytic transformation around a movable pole $z = z_0$ (see Prop.3.5), and the spaces of the initial conditions are obtained by the weighted blow-ups. In the present paper, the previous result is extended to a general quasi-homogeneous system (1.1). In Sec.4, our theory is applied to the first Painlevé hierarchy, which is a $2m$ -dimensional system of equations ($m = 1, 2, \dots$). The $2m$ -dimensional first Painlevé equation has m -types of Laurent series solutions. A complete list of the Kovalevskaya exponents of Laurent series solutions are given (Thm.4.1). Further, how to construct the space of initial conditions is demonstrated for the 4-dim first Painlevé equation.

2 Settings

2.1 Kovalevskaya exponent

Let us consider the system of differential equations

$$\frac{dx_i}{dz} = f_i(x_1, \dots, x_m, z) + g_i(x_1, \dots, x_m, z), \quad i = 1, \dots, m, \quad (2.1)$$

where f_i and g_i are polynomials in $(x_1, \dots, x_m, z) \in \mathbb{C}^{m+1}$. We suppose that

(A1) (f_1, \dots, f_m) is a quasi-homogeneous vector field satisfying

$$f_i(\lambda^{p_1} x_1, \dots, \lambda^{p_m} x_m, \lambda^r z) = \lambda^{p_i+1} f_i(x_1, \dots, x_m, z) \quad (2.2)$$

for any $\lambda \in \mathbb{C}$ and $i = 1, \dots, m$, where $(p_1, \dots, p_m, r) \in \mathbb{Z}_{>0}^{m+1}$ is a positive weight. The positive integers p_i and r are called the weighted degrees of x_i and z , respectively.

Put $f_i^A(x_1, \dots, x_m) := f_i(x_1, \dots, x_m, 0)$ and $f_i^N := f_i - f_i^A$ (i.e. f_i^A and f_i^N are autonomous and nonautonomous parts, respectively). Obviously they satisfy

$$\begin{aligned} f_i^A(\lambda^{p_1} x_1, \dots, \lambda^{p_m} x_m) &= \lambda^{p_i+1} f_i^A(x_1, \dots, x_m), \\ f_i^N(\lambda^{p_1} x_1, \dots, \lambda^{p_m} x_m, z) &= o(\lambda^{p_i+1}), \quad |\lambda| \rightarrow \infty. \end{aligned}$$

We assume that (g_1, \dots, g_m) is also small with respect to the above weight;

(A2) Suppose (g_1, \dots, g_m) satisfies

$$g_i(\lambda^{p_1}x_1, \dots, \lambda^{p_m}x_m, \lambda^r z) = o(\lambda^{p_i+1}), \quad |\lambda| \rightarrow \infty.$$

We also consider the truncated system

$$\frac{dx_i}{dz} = f_i^A(x_1, \dots, x_m), \quad i = 1, \dots, m. \quad (2.3)$$

Lemma 2.1. The truncated system is invariant under the scaling

$$(x_1, \dots, x_m, z) \mapsto (\lambda^{p_1}x_1, \dots, \lambda^{p_m}x_m, \lambda^{-1}z). \quad (2.4)$$

Further, if the equation

$$-p_i c_i = f_i^A(c_1, \dots, c_m), \quad i = 1, \dots, m \quad (2.5)$$

has a root $(c_1, \dots, c_m) \in \mathbb{C}^m$, $x_i(z) = c_i(z - z_0)^{-p_i}$ is an exact solution of the truncated system for any $z_0 \in \mathbb{C}$.

The variational equation of $dx_i/dz = f_i^A(x_1, \dots, x_m)$ along the solution $x_i(z) = c_i(z - z_0)^{-p_i}$ is given by

$$\frac{dy_i}{dz} = \sum_{k=1}^m \frac{\partial f_i^A}{\partial x_k}(c_1(z - z_0)^{-p_1}, \dots, c_m(z - z_0)^{-p_m}) y_k, \quad i = 1, \dots, m.$$

Substituting $y_i = \gamma_i(z - z_0)^{\lambda - p_i}$ with the aid of Eq.(2.2) provides

$$\sum_{k=1}^m \frac{\partial f_i^A}{\partial x_k}(c_1, \dots, c_m) \gamma_k + p_i \gamma_i = \lambda \gamma_i.$$

Hence, λ is an eigenvalue of the matrix $Df^A(c_1, \dots, c_m) + \text{diag}(p_1, \dots, p_m)$.

Definition 2.2. Fix a root $\{c_i\}_{i=1}^m$ of the equation $-p_i c_i = f_i^A(c_1, \dots, c_m)$. The matrix

$$K = \left\{ \frac{\partial f_i^A}{\partial x_j}(c_1, \dots, c_m) + p_i \delta_{ij} \right\}_{i,j=1}^m \quad (2.6)$$

and its eigenvalues are called the Kovalevskaya matrix and the Kovalevskaya exponents, respectively, of the system (2.1) associated with $\{c_i\}_{i=1}^m$.

Consider a formal series solution of Eq.(2.1) of the form

$$x_i = c_i(z - z_0)^{-p_i} + a_{i,1}(z - z_0)^{-p_i+1} + a_{i,2}(z - z_0)^{-p_i+2} + \dots \quad (2.7)$$

Coefficients $a_{i,j}$ are determined by substituting it into Eq.(2.1). The column vector $a_j = (a_{1,j}, \dots, a_{m,j})^T$ satisfies

$$(K - jI)a_j = (\text{a function of } c_i \text{ and } a_{i,k} \text{ with } k < j). \quad (2.8)$$

If a positive integer j is not an eigenvalue of K , a_j is uniquely determined. If a positive integer j is an eigenvalue of K and (2.8) has no solutions, we have to introduce a logarithmic term $\log(z - z_0)$ into the coefficient a_j . In this case, the system (2.1) has no Laurent series solution of the form (2.7) with a given $\{c_i\}_{i=1}^m$. If a positive integer j is an eigenvalue of K and (2.8) has a solution a_j , then $a_j + v$ is also a solution for any eigenvectors v . This implies that the series solution (2.7) includes a free parameter in $(a_{1,j}, \dots, a_{m,j})$. If it includes $k - 1$ free parameters other than z_0 , (2.7) represents a k -parameter family of Laurent series solutions. Hence, the classical Painlevé test [1, 14] for the necessary condition for the Painlevé property is stated as follows;

Classical Painlevé test. If the system (2.1) satisfying (A1) and (A2) has the Painlevé property in the sense that any solutions are meromorphic, then there exist numbers $\{c_i\}_{i=1}^m$ such that all Kovalevskaya exponents except for -1 (see below) are positive integers, and the Kovalevskaya matrix is semisimple. In this case, (2.7) represents an m -parameter family of Laurent series solutions.

In Prop.3.5, we will give a necessary and sufficient condition for the series (2.7) to be a convergent Laurent series. The next lemmas are well known [2, 9].

Lemma 2.3. (i) $\lambda = -1$ is always a Kovalevskaya exponent with the eigenvector $(-p_1 c_1, \dots, -p_m c_m)^T$.
(ii) $\lambda = 0$ is a Kovalevskaya exponent associated with $\{c_i\}_{i=1}^m$ if and only if $\{c_i\}_{i=1}^m$ is not an isolated root of the equation $-p_i c_i = f_i^A(c_1, \dots, c_m)$.

Lemma 2.4. Consider the Hamiltonian system

$$\frac{dx_i}{dz} = -\frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dz} = \frac{\partial H}{\partial x_i}, \quad (i = 1, \dots, m) \quad (2.9)$$

with a holomorphic Hamiltonian satisfying

$$H(\lambda^{p_1} x_1, \lambda^{q_1} y_1, \dots, \lambda^{p_m} x_m, \lambda^{q_m} y_m) = \lambda^{h+1} H(x_1, y_1, \dots, x_m, y_m). \quad (2.10)$$

If λ is a Kovalevskaya exponent, so is μ given by $\lambda + \mu = h$.

In what follows, a Kovalevskaya exponent is called a K-exponent for simplicity. Let us consider the system (2.1) with the assumptions (A1) and (A2). We show that the K-exponents are invariant under a certain class of coordinate transformations.

Consider a holomorphic transformation

$$x_i = \varphi_i(y_1, \dots, y_m), \quad (i = 1, \dots, m), \quad (2.11)$$

which is locally biholomorphic near the point $(x_1, \dots, x_m) = (c_1, \dots, c_m)$. The inverse transformation is denoted by $y_i = \psi_i(x_1, \dots, x_m)$. Suppose that φ_i satisfies

$$\varphi_i(\lambda^{q_1} y_1, \dots, \lambda^{q_m} y_m) = \lambda^{p_i} \varphi_i(y_1, \dots, y_m), \quad \lambda \in \mathbb{C}, \quad (2.12)$$

with some $(q_1, \dots, q_m) \in \mathbb{Z}^m$ for a given (p_1, \dots, p_m) in (A1). It is easy to see that the inverse satisfies

$$\psi_i(\lambda^{p_1} x_1, \dots, \lambda^{p_m} x_m) = \lambda^{q_i} \psi_i(x_1, \dots, x_m).$$

By the transformation, Eq.(2.1) is brought into the new system

$$\frac{dy_i}{dz} = \sum_{j=1}^m (D\varphi)_{ij}^{-1} f_j(\varphi(y), z) + \sum_{j=1}^m (D\varphi)_{ij}^{-1} g_j(\varphi(y), z) =: F_i(y, z) + G_i(y, z), \quad (2.13)$$

where $y = (y_1, \dots, y_m)$ and $\varphi = (\varphi_1, \dots, \varphi_m)$. It is straightforward to show that the new system satisfies the conditions (A1) and (A2), in which (p_1, \dots, p_m) is replaced by (q_1, \dots, q_m) . Hence, the K-exponents of (2.13) with the weight (q_1, \dots, q_m) are well defined.

Theorem 2.5. The K-exponents of the system (2.13) coincide with those of (2.1).

Proof. Differentiated by y_k , Eq.(2.12) yields

$$\frac{\partial \varphi_i}{\partial y_k}(\lambda^{q_1} y_1, \dots, \lambda^{q_m} y_m) = \lambda^{p_i - q_k} \frac{\partial \varphi_i}{\partial y_k}(y_1, \dots, y_m). \quad (2.14)$$

Differentiating in λ and putting $\lambda = 1$ for Eqs.(2.12) and (2.14), we obtain

$$\sum_{k=1}^m \frac{\partial \varphi_i}{\partial y_k} q_k y_k = p_i \varphi_i(y_1, \dots, y_m), \quad (2.15)$$

$$\sum_{l=1}^m \frac{\partial^2 \varphi_i}{\partial y_k \partial y_l} q_l y_l = (p_i - q_k) \frac{\partial \varphi_i}{\partial y_k}(y_1, \dots, y_m). \quad (2.16)$$

The (i, k) -component of the Kovalevskaya matrix \tilde{K} of (2.13) is given by

$$\tilde{K}_{ik} = \sum_{j=1}^m \frac{\partial (D\varphi)_{ij}^{-1}}{\partial y_k} f_j^A(\varphi(y)) + \sum_{j=1}^m (D\varphi)_{ij}^{-1} \sum_{l=1}^m \frac{\partial f_j^A}{\partial y_l}(\varphi(y)) \frac{\partial \varphi_l}{\partial y_k}(y) + q_i \delta_{ik}.$$

By using the equalities (2.15), (2.16) and $-p_i c_i = f_i^A(c_1, \dots, c_m)$, we can show that \tilde{K}_{ik} is rewritten as

$$\tilde{K}_{ik} = \sum_{j,l=1}^m (D\varphi)_{ij}^{-1} (p_j \delta_{jl} + (Df^A)_{jl}) (D\varphi)_{lk}.$$

This proves that \tilde{K} is similar to K . \square

Proposition 2.6. If the system (2.1) has a formal series solution (2.7), it is a convergent series on $0 < |z - z_0| < \varepsilon$ for some $\varepsilon > 0$.

In Sec.3, a formal series solution (2.7) is regarded as an integral curve on an unstable manifold of a certain vector field. Then, Prop.2.6 immediately follows from the unstable manifold theorem, see also Goriely [8] for the same result for autonomous systems.

Next, let us consider the series solution of the form

$$x_i(z) = c_i(z - z_0)^{-q_i} + \sum_{n=1}^{\infty} a_{i,n}(z - z_0)^{-q_i+n}, \quad (i = 1, \dots, m). \quad (2.17)$$

Note that the order of the leading term is q_i , not p_i .

Proposition 2.7. If $0 \leq q_i < p_i$ and $c_i \neq 0$ for any i , then $q_i = 0$ for all i .

Proposition 2.8. For the system (2.1), we further suppose the following condition.

(S) A fixed point of the truncated system is only the origin, i.e,

$$f_i^A(x_1, \dots, x_m) = 0 \quad (i = 1, \dots, m) \Rightarrow (x_1, \dots, x_m) = (0, \dots, 0). \quad (2.18)$$

If $q_i > p_i$ for some i , then $c_i = 0$.

Prop.2.7 means that if the order of a pole of $x_i(z)$ is smaller than p_i for all $i = 1, \dots, m$, then (2.17) should be a local analytic solution. Prop.2.8 implies that there are no Laurent series solutions $x_i(z)$ whose pole order is larger than p_i . Proofs of Prop.2.7 and 2.8 are given in Appendix B. Combining three propositions, we have

Theorem 2.9. If the system (2.1) satisfies (A1), (A2) and (S), any formal series solutions with a singularity at $z = z_0$ are of the form (2.7) such that $(c_1, \dots, c_m) \neq (0, \dots, 0)$, and they are convergent.

Remark 2.10. If the truncated system has a fixed point other than the origin, then it has a family of fixed points which forms an algebraic curve on \mathbb{C}^m due to the quasi-homogeneity. If the truncated system is a Hamiltonian system with the Hamiltonian function $H(x_1, \dots, x_m)$, the assumption (S) implies that a singularity of the algebraic variety defined by $\{H = 0\}$ is isolated. This fact will be essentially used to study a relationship between the Painlevé equations and singularity theory.

Due to (A1), the truncated system (2.3) is invariant under the \mathbb{Z}_s action

$$(x_1, \dots, x_m, z) \mapsto (\omega^{p_1} x_1, \dots, \omega^{p_m} x_m, \omega^r z), \quad \omega := e^{2\pi i/s}, \quad (2.19)$$

if $s = r + 1$. For later purpose, we assume that the full system (2.1) is also invariant under the same action;

(A3) The system (2.1) is invariant under the \mathbb{Z}_s action (2.19) with $s = r + 1$.

Example 2.11. The first, second and fourth Painlevé equations in Hamiltonian

forms are given by

$$(P_I) \begin{cases} \frac{dx}{dz} = 6y^2 + z \\ \frac{dy}{dz} = x, \end{cases} \quad (2.20)$$

$$(P_{II}) \begin{cases} \frac{dx}{dz} = 2y^3 + yz + \alpha \\ \frac{dy}{dz} = x, \end{cases} \quad (2.21)$$

$$(P_{IV}) \begin{cases} \frac{dx}{dz} = -x^2 + 2xy + 2xz + \alpha \\ \frac{dy}{dz} = -y^2 + 2xy - 2yz + \beta, \end{cases} \quad (2.22)$$

where $\alpha, \beta \in \mathbb{C}$ are arbitrary parameters. These systems satisfy the assumptions (A1) to (A3) with the weights

$$\begin{aligned} (P_I) \quad & (p_1, p_2, r, s) = (3, 2, 4, 5), \\ (P_{II}) \quad & (p_1, p_2, r, s) = (2, 1, 2, 3), \\ (P_{IV}) \quad & (p_1, p_2, r, s) = (1, 1, 1, 2), \end{aligned}$$

where $f = (6y^2 + z, x)$, $g = (0, 0)$ for (P_I) , $f = (2y^3 + yz, x)$, $g = (\alpha, 0)$ for (P_{II}) and $f = (-x^2 + 2xy + 2xz, -y^2 + 2xy - 2yz)$, $g = (\alpha, \beta)$ for (P_{IV}) . In Chiba [3], these systems are investigated by means of the weighted projective spaces $\mathbb{C}P^3(p_1, p_2, r, s)$. One of the purposes in this paper is to extend the previous result to a general system (2.1). For a given weight (p_1, \dots, p_m, r) satisfying (A1) to (A3), the weighted projective space $\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s)$ gives a suitable compactification of \mathbb{C}^{m+1} (the space of the dependent variables and the independent variable), which is effective to investigate the asymptotic behavior of solutions of the system.

Remark 2.12. A few remarks are in order. If $f = (f_1, \dots, f_m)$ is independent of z (i.e. $f_i^N = 0$), we define $r = 0$ and $s = 1$. In this case, we need not assume (A3); the action is trivial. All results in this paper hold even in this case. Some of our analysis is still valid even if (p_1, \dots, p_m) includes zeros or negative integers. See [4] for the detail, in which the third, fifth and sixth Painlevé equations are treated. In general, a weight (p_1, \dots, p_m, r) satisfying (A1) to (A3) is not unique. All possible weights can be calculated through the Newton diagram of the system (2.1), see [3]. In this paper, we fix one of the weights.

2.2 Weighted projective space

Consider the weighted \mathbb{C}^* -action on \mathbb{C}^{m+2} defined by

$$(x_1, \dots, x_m, z, \varepsilon) \mapsto (\lambda^{p_1} x_1, \dots, \lambda^{p_m} x_m, \lambda^r z, \lambda^s \varepsilon), \quad \lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\},$$

where the weights (p_1, \dots, p_m, r, s) are relatively prime positive integers. The quotient space

$$\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s) := \mathbb{C}^{m+2} \setminus \{0\} / \mathbb{C}^*$$

gives an $m + 1$ dimensional orbifold called the weighted projective space.

In order to show that a weighted projective space is indeed an orbifold, we will introduce the inhomogeneous coordinates. For simplicity, we demonstrate it for a three dimensional space $\mathbb{C}P^3(p, q, r, s)$.

The space $\mathbb{C}P^3(p, q, r, s)$ is defined by the equivalence relation on $\mathbb{C}^4 \setminus \{0\}$

$$(x, y, z, \varepsilon) \sim (\lambda^p x, \lambda^q y, \lambda^r z, \lambda^s \varepsilon).$$

(i) When $x \neq 0$,

$$(x, y, z, \varepsilon) \sim (1, x^{-q/p}y, x^{-r/p}z, x^{-s/p}\varepsilon) =: (1, Y_1, Z_1, \varepsilon_1).$$

Due to the choice of the branch of $x^{1/p}$, we also obtain

$$(Y_1, Z_1, \varepsilon_1) \sim (e^{-2q\pi i/p}Y_1, e^{-2r\pi i/p}Z_1, e^{-2s\pi i/p}\varepsilon_1),$$

by putting $x \mapsto e^{2\pi i}x$. This implies that the subset of $\mathbb{C}P^3(p, q, r, s)$ such that $x \neq 0$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_p$, where the \mathbb{Z}_p -action is defined as above.

(ii) When $y \neq 0$,

$$(x, y, z, \varepsilon) \sim (y^{-p/q}x, 1, y^{-r/q}z, y^{-s/q}\varepsilon) =: (X_2, 1, Z_2, \varepsilon_2).$$

Because of the choice of the branch of $y^{1/q}$, we obtain

$$(X_2, Z_2, \varepsilon_2) \sim (e^{-2p\pi i/q}X_2, e^{-2r\pi i/q}Z_2, e^{-2s\pi i/q}\varepsilon_2).$$

Hence, the subset of $\mathbb{C}P^3(p, q, r, s)$ with $y \neq 0$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_q$.

(iii) When $z \neq 0$,

$$(x, y, z, \varepsilon) \sim (z^{-p/r}x, z^{-q/r}y, 1, z^{-s/r}\varepsilon) =: (X_3, Y_3, 1, \varepsilon_3).$$

Similarly, the subset $\{z \neq 0\} \subset \mathbb{C}P^3(p, q, r, s)$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_r$.

(iv) When $\varepsilon \neq 0$,

$$(x, y, z, \varepsilon) \sim (\varepsilon^{-p/s}x, \varepsilon^{-q/s}y, \varepsilon^{-r/s}z, 1) =: (X_4, Y_4, Z_4, 1).$$

The subset $\{\varepsilon \neq 0\} \subset \mathbb{C}P^3(p, q, r, s)$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_s$.

This proves that the orbifold structure of $\mathbb{C}P^3(p, q, r, s)$ is given by

$$\mathbb{C}P^3(p, q, r, s) = \mathbb{C}^3/\mathbb{Z}_p \cup \mathbb{C}^3/\mathbb{Z}_q \cup \mathbb{C}^3/\mathbb{Z}_r \cup \mathbb{C}^3/\mathbb{Z}_s.$$

The local charts $(Y_1, Z_1, \varepsilon_1)$, $(X_2, Z_2, \varepsilon_2)$, $(X_3, Y_3, \varepsilon_3)$ and (X_4, Y_4, Z_4) defined above are called inhomogeneous coordinates as the usual projective space. Note that they give coordinates on the lift \mathbb{C}^3 , not on the quotient $\mathbb{C}^3/\mathbb{Z}_i$ ($i = p, q, r, s$). Therefore,

any equations written in these inhomogeneous coordinates should be invariant under the corresponding \mathbb{Z}_i actions.

The transformations between inhomogeneous coordinates are give by

$$\begin{cases} X_4 = \varepsilon_1^{-p/s} = X_2 \varepsilon_2^{-p/s} = X_3 \varepsilon_3^{-p/s} \\ Y_4 = Y_1 \varepsilon_1^{-q/s} = \varepsilon_2^{-q/s} = Y_3 \varepsilon_3^{-q/s} \\ Z_4 = Z_1 \varepsilon_1^{-r/s} = Z_2 \varepsilon_2^{-r/s} = \varepsilon_3^{-r/s}. \end{cases} \quad (2.23)$$

An extension to the $m + 1$ dimensional case $\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s)$ is straightforward. The orbifold structure is characterized by

$$\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s) = \mathbb{C}^{m+1}/\mathbb{Z}_{p_1} \cup \dots \cup \mathbb{C}^{m+1}/\mathbb{Z}_{p_m} \cup \mathbb{C}^{m+1}/\mathbb{Z}_r \cup \mathbb{C}^{m+1}/\mathbb{Z}_s.$$

The inhomogeneous coordinates are defined as above on each chart. In what follows, we use the notation (x_1, \dots, x_m, z) for the inhomogeneous coordinates of the local chart $\mathbb{C}^{m+1}/\mathbb{Z}_s$ because a system of differential equations will be given on this chart. For example, the transformation between the inhomogeneous coordinates (x_1, \dots, x_m, z) on $\mathbb{C}^{m+1}/\mathbb{Z}_s$ and the j -th inhomogeneous coordinates $(X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_m, Z, \varepsilon)$ on $\mathbb{C}^{m+1}/\mathbb{Z}_{p_j}$ is give by

$$x_i = X_i \varepsilon^{-p_i/s} \ (i \neq j), \ x_j = \varepsilon^{-p_j/s}, \ z = Z \varepsilon^{-r/s}. \quad (2.24)$$

Hence, the subset $\{\varepsilon = 0\}$ on $\mathbb{C}^{m+1}/\mathbb{Z}_{p_j}$ is attached at “infinity” of the chart $\mathbb{C}^{m+1}/\mathbb{Z}_s$.

3 A differential equation on a weighted projective space

Now we give the system of polynomial differential equations

$$\frac{dx_i}{dz} = f_i(x_1, \dots, x_m, z) + g_i(x_1, \dots, x_m, z), \quad i = 1, \dots, m, \quad (3.1)$$

satisfying (A1), (A2) and (A3) on the (x_1, \dots, x_m, z) coordinates of the space $\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s)$. Note that the inhomogeneous coordinates (x_1, \dots, x_m, z) are coordinates for the lift \mathbb{C}^{m+1} of the quotient $\mathbb{C}^{m+1}/\mathbb{Z}_s \subset \mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s)$. Since the system (3.1) is invariant under the \mathbb{Z}_s action (2.19), it is well defined on the quotient space $\mathbb{C}^{m+1}/\mathbb{Z}_s$.

Let us express it on the j -th local chart $\mathbb{C}^{m+1}/\mathbb{Z}_{p_j}$ by the transformation (2.24). Due to the assumptions, we have

$$\begin{aligned} & f_i(X_1 \varepsilon^{-p_1/s}, \dots, \varepsilon^{-p_j/s}, \dots, X_m \varepsilon^{-p_m/s}, Z \varepsilon^{-r/s}) \\ &= \varepsilon^{-(1+p_i)/s} f_1(X_1, \dots, 1, \dots, X_m, Z), \\ & g_i(X_1 \varepsilon^{-p_1/s}, \dots, \varepsilon^{-p_j/s}, \dots, X_m \varepsilon^{-p_m/s}, Z \varepsilon^{-r/s}) \\ &= \varepsilon^{1-(1+p_i)/s} \times (\text{a polynomial of } X_1, \dots, X_m, Z, \varepsilon). \end{aligned}$$

With the aid of these equalities, (3.1) is written on the $(X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_m, Z, \varepsilon)$ coordinates of $\mathbb{C}^{m+1}/\mathbb{Z}_{p_j}$ as

$$\begin{cases} \frac{dX_i}{d\varepsilon} = \frac{1}{s\varepsilon} \left(p_i X_i - p_j \frac{f_i + \varepsilon G_i}{f_j + \varepsilon G_j} \right), & (i = 1, \dots, m; i \neq j), \\ \frac{dZ}{d\varepsilon} = \frac{1}{s\varepsilon} \left(rZ - \frac{p_j \varepsilon}{f_j + \varepsilon G_j} \right), \end{cases} \quad (3.2)$$

where $f_i = f_i(X_1, \dots, 1, \dots, X_m, Z)$ (the unity is substituted at the j -th argument) and G_i is a polynomial in $(X_1, \dots, X_m, Z, \varepsilon)$ determined by g_i . This system is rational and invariant under the \mathbb{Z}_{p_j} action despite the fact that the coordinate transformation (2.24) is not rational.

Proposition 3.1. Give the system (3.1) on the (x_1, \dots, x_m, z) -coordinates of the local chart $\mathbb{C}^{m+1}/\mathbb{Z}_s$ of $\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s)$. If the system satisfies the assumptions (A1) to (A3), it induces a well defined rational differential equations on $\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s)$.

Example 3.2. We give the first Painlevé equation $x' = 6y^2 + z$, $y' = x$ on the fourth local chart (x, y, z) of $\mathbb{C}P^3(3, 2, 4, 5)$. By (2.23), it is transformed into the following equations

$$\begin{aligned} \frac{dY_1}{d\varepsilon_1} &= \frac{3 - 12Y_1^3 - 2Y_1Z_1}{\varepsilon_1(-30Y_1^2 - 5Z_1)}, & \frac{dZ_1}{d\varepsilon_1} &= \frac{3\varepsilon_1 - 24Y_1^2Z_1 - 4Z_1^2}{\varepsilon_1(-30Y_1^2 - 5Z_1)}, \\ \frac{dX_2}{d\varepsilon_2} &= \frac{-12 - 2Z_2 + 3X_2^2}{5X_2\varepsilon_2}, & \frac{dZ_2}{d\varepsilon_2} &= \frac{-2\varepsilon_2 + 4X_2Z_2}{5X_2\varepsilon_2}, \\ \frac{dX_3}{d\varepsilon_3} &= \frac{24Y_3^2 + 4 - 3X_3\varepsilon_3}{-5\varepsilon_3^2}, & \frac{dY_3}{d\varepsilon_3} &= \frac{4X_3 - 2Y_3\varepsilon_3}{-5\varepsilon_3^2}, \end{aligned}$$

on the other inhomogeneous coordinates. Although the transformations (2.23) have branches, the above equations are rational because the first Painlevé equation satisfies (A1) to (A3) with $(p, q, r, s) = (3, 2, 4, 5)$. Hence, they define a rational ODE on $\mathbb{C}P^3(3, 2, 4, 5)$ in the sense of an orbifold.

It is convenient to rewrite (3.2) as an autonomous vector field of the form

$$\begin{cases} \frac{dX_i}{dt} = p_i X_i - p_j \frac{f_i + \varepsilon G_i}{f_j + \varepsilon G_j} & (i = 1, \dots, m; i \neq j), \\ \frac{dZ}{dt} = rZ - \frac{p_j \varepsilon}{f_j + \varepsilon G_j}, \\ \frac{d\varepsilon}{dt} = s\varepsilon. \end{cases} \quad (3.3)$$

The new independent variable t parameterizes integral curves of (3.2). Note that how to rewrite the system (3.2) as an autonomous vector field is not unique. To

construct the space of initial conditions, rewriting as a polynomial vector field

$$\begin{cases} \frac{dX_i}{dt} = p_i X_i (f_j + \varepsilon G_j) - p_j (f_i + \varepsilon G_i), & (i = 1, \dots, m; i \neq j), \\ \frac{dZ}{dt} = r Z (f_j + \varepsilon G_j) - p_j \varepsilon, \\ \frac{d\varepsilon}{dt} = s \varepsilon (f_j + \varepsilon G_j), \end{cases} \quad (3.4)$$

may be more convenient, though in this section we will use the form (3.3).

Let us investigate the K-exponents of the system (3.1). Let (c_1, \dots, c_m) be one of the roots of the equation $-p_i c_i = f_i^A(c_1, \dots, c_m)$, and consider a series solution (2.7). If it is not a local holomorphic solution, $(c_1, \dots, c_m) \neq (0, \dots, 0)$ due to Prop.2.7. Assume $c_j \neq 0$. On the $(X_1, \dots, X_m, Z, \varepsilon)$ coordinates of $\mathbb{C}^{m+1}/\mathbb{Z}_{p_j}$, we obtain

$$\begin{cases} X_i = x_i x_j^{-p_i/p_j} = c_i c_j^{-p_i/p_j} (1 + O(z - z_0)), \\ Z = z x_j^{-r/p_j} = z c_j^{-r/p_j} (z - z_0)^r (1 + O(z - z_0)), \\ \varepsilon = x_j^{-s/p_j} = c_j^{-s/p_j} (z - z_0)^s (1 + O(z - z_0)). \end{cases}$$

In particular,

$$X_i \rightarrow c_i c_j^{-p_i/p_j} \quad (i \neq j), \quad Z, \varepsilon \rightarrow 0,$$

as $z \rightarrow z_0$.

Lemma 3.3. The point

$$(X_1, \dots, X_m, Z, \varepsilon) = (c_1 c_j^{-p_1/p_j}, \dots, c_m c_j^{-p_m/p_j}, 0, 0) \quad (3.5)$$

is a fixed point of the vector field (3.3).

Proof. A fixed point of (3.3) satisfying $Z = \varepsilon = 0$ is given as a root of

$$p_i X_i f_j^A(X_1, \dots, 1, \dots, X_m) - p_j f_i^A(X_1, \dots, 1, \dots, X_m) = 0, \quad (i \neq j).$$

At the point (3.5), the left hand side is estimated as

$$\begin{aligned} & p_i c_i c_j^{-p_i/p_j} f_j^A(c_1 c_j^{-p_1/p_j}, \dots, c_m c_j^{-p_m/p_j}) - p_j f_i^A(c_1 c_j^{-p_1/p_j}, \dots, c_m c_j^{-p_m/p_j}) \\ &= p_i c_i c_j^{-p_i/p_j} c_j^{-(1+p_j)/p_j} f_j^A(c_1, \dots, c_m) - p_j c_j^{-(1+p_i)/p_j} f_i^A(c_1, \dots, c_m). \end{aligned}$$

Then, the equality $-p_i c_i = f_i^A(c_1, \dots, c_m)$ proves that the above quantity actually becomes zero. \square

This lemma suggests that the behavior of the series solution (2.7) as $z \rightarrow z_0$ is governed by local properties of the fixed point (3.5). In particular, the dynamical systems theory will be applied to the study of the fixed point. Due to the orbifold structure, the inhomogeneous coordinates $(X_1, \dots, X_m, Z, \varepsilon)$ should be divided by the \mathbb{Z}_{p_j} -action (Sec.2.3). Hence, all points expressed as (3.5) obtained by different choices of roots c_j^{-1/p_j} represent the same point on the quotient space $\mathbb{C}^{m+1}/\mathbb{Z}_{p_j}$.

3.1 Kovalevskaya exponents

The main theorem in this section is stated as follows.

Theorem 3.4. The eigenvalues of the Jacobi matrix of the vector field (3.3) at the fixed point (3.5) are given by r, s and K-exponents of the system (3.1) other than the trivial exponent -1 . (If we use the polynomial vector field (3.4) instead of (3.3), eigenvalues change by a constant factor).

Proof. We assume $j = m$ for simplicity. Thus, (3.3) is an equation for $(X_1, \dots, X_{m-1}, Z, \varepsilon)$.

Let $v = (-p_1 c_1, \dots, -p_m c_m)^T$ be the eigenvector of the K-matrix associated with the eigenvalue -1 (see Lemma 2.3). Put $v_1 = (-p_1 c_1, \dots, -p_{m-1} c_{m-1})^T$, $v_2 = -p_m c_m$ and

$$P = \begin{pmatrix} I & v_1 \\ 0 & v_2 \end{pmatrix},$$

where I denotes the $(m-1) \times (m-1)$ identity matrix. We obtain

$$P^{-1} K P = \begin{pmatrix} \tilde{K} & 0 \\ * & -1 \end{pmatrix},$$

where \tilde{K} is an $(m-1) \times (m-1)$ matrix whose eigenvalues are K-exponents other than -1 . The (i, j) component of \tilde{K} is given by

$$\tilde{K}_{i,j} = p_i \delta_{ij} + \frac{\partial f_i^A}{\partial x_j}(c_1, \dots, c_m) - \frac{p_i c_i}{p_m c_m} \frac{\partial f_m^A}{\partial x_j}(c_1, \dots, c_m).$$

On the other hand, the Jacobi matrix of (3.3) at the fixed point is of the form

$$J = \left(\begin{array}{c|cc} \tilde{J} & * & * \\ \hline 0 & r & * \\ 0 & 0 & s \end{array} \right), \quad (3.6)$$

where \tilde{J} is an $(m-1) \times (m-1)$ matrix whose (i, j) component is given by

$$\tilde{J}_{i,j} = p_i \delta_{ij} - \frac{p_m}{f_m^A} \frac{\partial f_i^A}{\partial x_j} + p_m \frac{f_i^A}{(f_m^A)^2} \frac{\partial f_m^A}{\partial x_j}, \quad (3.7)$$

where f_i^A is estimated at the point $(c_1 c_m^{-p_1/p_m}, \dots, c_{m-1} c_m^{-p_{m-1}/p_m}, 1)$. Due to the quasi-homogeneity and the equality $-p_i c_i = f_i^A(c_1, \dots, c_m)$, we have

$$\begin{aligned} f_i^A(c_1 c_m^{-p_1/p_m}, \dots, c_{m-1} c_m^{-p_{m-1}/p_m}, 1) &= c_m^{-(1+p_i)/p_m} f_i^A(c_1, \dots, c_m) = -p_i c_i c_m^{-(1+p_i)/p_m}, \\ \frac{\partial f_i^A}{\partial x_j}(c_1 c_m^{-p_1/p_m}, \dots, c_{m-1} c_m^{-p_{m-1}/p_m}, 1) &= c_m^{-(p_i+1-p_j)/p_m} \frac{\partial f_i^A}{\partial x_j}(c_1, \dots, c_m). \end{aligned}$$

By a suitable scaling of the independent variable z of the original system (3.1), we can assume without loss of generality that $c_m = 1$. Substituting the above equalities into (3.7) with $c_m = 1$, we obtain $\tilde{J}_{i,j} = \tilde{K}_{i,j}$. \square

Since eigenvalues of the Jacobi matrix are invariant under the actions of diffeomorphisms, the K-exponents are invariant under a wide class of coordinate transformations. In particular, the set of all K-exponents associated with all roots $\{c_i\}_{i=1}^m$ are invariant under the automorphisms on $\mathbb{C}P^{m+1}(p_1, \dots, p_m, r, s)$.

3.2 Extended Painlevé test

Let $\lambda_2, \dots, \lambda_m, r, s$ be eigenvalues of the Jacobi matrix J of (3.3) at the fixed point (3.5), among which $\lambda_2, \dots, \lambda_m$ are K-exponents of (3.1) ($\lambda_1 = -1$ is used for the trivial one). Put $\hat{X}_i = X_i - c_i c_j^{-p_i/p_j}$. Then, Eq.(3.3) is rewritten as

$$\frac{d\hat{\mathbf{X}}}{dt} = J\hat{\mathbf{X}} + F(\hat{\mathbf{X}}), \quad \hat{\mathbf{X}} = (\hat{X}_1, \dots, \hat{X}_{j-1}, \hat{X}_{j+1}, \dots, \hat{X}_m, Z, \varepsilon), \quad (3.8)$$

with the nonlinearity F . Suppose that $\text{Re}(\lambda_i) \leq 0$ for $i = 2, \dots, k$, and $\text{Re}(\lambda_i) > 0$ for $i = k+1, \dots, m$. Since r and s are positive integers, J has exactly $m - k + 2$ eigenvalues with positive real parts. Thus, the system (3.8) has an $m - k + 2$ dimensional unstable manifold at the origin; by a suitable linear transformation, (3.8) is rewritten as

$$\begin{cases} \frac{d\mathbf{X}_u}{dt} = J_u \mathbf{X}_u + F_u(\mathbf{X}_u, \mathbf{X}_s), & \mathbf{X}_u \in \mathbb{C}^{m-k+2}, \\ \frac{d\mathbf{X}_s}{dt} = J_s \mathbf{X}_s + F_s(\mathbf{X}_u, \mathbf{X}_s), & \mathbf{X}_s \in \mathbb{C}^{k-1}, \end{cases}$$

where real parts of eigenvalues of J_u and J_s are positive and nonpositive, respectively. Due to the unstable manifold theorem, there exists a local analytic function φ satisfying $\varphi(0) = D\varphi(0) = 0$ such that the set $(\mathbf{X}_u, \varphi(\mathbf{X}_u))$ expresses the unstable manifold. Then,

$$\frac{d\mathbf{X}_u}{dt} = J_u \mathbf{X}_u + F_u(\mathbf{X}_u, \varphi(\mathbf{X}_u)) \quad (3.9)$$

gives the dynamics on the unstable manifold. The purpose in this section is to prove

Proposition 3.5. The system (3.1) has an $m - k + 1$ parameter family of convergent Laurent series solutions of the form

$$x_i(z) = c_i(z - z_0)^{-p_i} + \sum_{n=1}^{\infty} a_{i,n}(z - z_0)^{-p_i+n}, \quad i = 1, \dots, m,$$

where $\{a_{i,n}\}$ includes $m - k$ free parameters other than z_0 , if and only if

- (i) $\lambda_{k+1}, \dots, \lambda_m$ are positive integers,
- (ii) J_u is semi-simple, and

(iii) the system (3.9) on the unstable manifold is linearizable by a local analytic transformation.

In particular, the system (3.1) has an m parameter family of Laurent series solutions if and only if

- (i) all K-exponents except for $\lambda_1 = -1$ are positive integers (classical Painlevé test),
- (ii) the Jacobi matrix J is semisimple, and
- (iii) the system (3.8) is linearizable by a local analytic transformation.

A similar result is also obtained by Goriely [8] for autonomous systems. A linearization of the system (3.9) is achieved by Poincaré-Dulac normal form theory by finite steps. For the convenience of the reader, a brief review of the normal form theory is given in Appendix A. In [3, 4], it is proved that the first to sixth Painlevé equations are linearizable.

Proof of Prop.3.5. By the standard perturbation method (the variation of constants method), a general solution of (3.8) is expressed as

$$\begin{aligned} X_i(t) &= c_i c_j^{-p_i/p_j} + h_i(\alpha_2 e^{\lambda_2 t}, \dots, \alpha_m e^{\lambda_m t}, z_0 e^{rt}, \varepsilon_0 e^{st}), \quad (i = 1, \dots, m; i \neq j), \\ Z(t) &= z_0 e^{rt} + c_j^{1/p_j} \varepsilon_0 e^{st} + e^{st} h_{m+1}(\alpha_2 e^{\lambda_2 t}, \dots, \alpha_m e^{\lambda_m t}, z_0 e^{rt}, \varepsilon_0 e^{st}), \end{aligned}$$

and $\varepsilon(t) = \varepsilon_0 e^{st}$, where $\alpha_2, \dots, \alpha_m, z_0, \varepsilon_0$ are free parameters determined by an initial condition. The functions h_1, \dots, h_{m+1} are *formal* power series with $h_i(0) = 0$ whose coefficients are polynomial in t . More precisely, they are expressed as

$$h_i(\alpha_2 e^{\lambda_2 t}, \dots, \varepsilon_0 e^{st}) = \sum_{|n|=1}^{\infty} h_{i,n}(t) e^{\langle \lambda, n \rangle t}, \quad (i = 1, \dots, m+1; i \neq j),$$

where $n = (n_2, \dots, n_{m+2})$, $|n| = n_2 + \dots + n_{m+2}$ and $\langle \lambda, n \rangle = \lambda_2 n_2 + \dots + \lambda_m n_m + r n_{m+1} + s n_{m+2}$. The function $h_{i,n}(t)$ is polynomial in t . Moving to the original coordinates (x_1, \dots, x_m, z) , we obtain the next lemma, which can be also proved directly from Eq.(3.1).

Lemma 3.6. The system (3.1) has a *formal* series solution of the form

$$x_i(z) = c_i T^{-p_i} (1 + \tilde{h}_i(\alpha_2 T^{\lambda_2}, \dots, \alpha_m T^{\lambda_m}, z_0 T^r, \varepsilon_0 T^s)), \quad T := z - z_0, \quad (3.10)$$

where \tilde{h}_i is a formal power series in the arguments, whose coefficients are polynomial in $\log T$, and $\alpha_2, \dots, \alpha_m, z_0, \varepsilon_0$ are free parameters.

A solution on the unstable manifold is obtained by putting $\alpha_2 = \dots = \alpha_k = 0$;

$$\begin{aligned} X_i(t) &= c_i c_j^{-p_i/p_j} + h_i(0, \dots, 0, \alpha_{k+1} e^{\lambda_{k+1} t}, \dots, \alpha_m e^{\lambda_m t}, z_0 e^{rt}, \varepsilon_0 e^{st}), \\ Z(t) &= z_0 e^{rt} + c_j^{1/p_j} \varepsilon_0 e^{st} + e^{st} h_{m+1}(0, \dots, 0, \alpha_{k+1} e^{\lambda_{k+1} t}, \dots, \alpha_m e^{\lambda_m t}, z_0 e^{rt}, \varepsilon_0 e^{st}), \end{aligned}$$

This is a *convergent* series solution because of the unstable manifold theorem. Moving to the original coordinates (x_1, \dots, x_m, z) , we obtain

$$x_i(z) = c_i T^{-p_i} (1 + \tilde{h}_i(0, \dots, 0, \alpha_{k+1} T^{\lambda_{k+1}}, \dots, \alpha_m T^{\lambda_m}, z_0 T^r, \varepsilon_0 T^s)), \quad (3.11)$$

where the right hand side is a convergent power series in the arguments, whose coefficients are polynomial in $\log T$, and $\alpha_{k+1}, \dots, \alpha_m, z_0, \varepsilon_0$ are $m - k + 2$ free parameters. On the parameter space, there are curves $(\alpha_{k+1}(t), \dots, \alpha_m(t), z_0(t), \varepsilon_0(t))$, on which the above solution represents the same solution. Hence, (3.11) defines an $m - k + 1$ parameter family of solutions. It does not include $\log T$ if and only if the coefficients of h_i do not include polynomial in t . Then, Prop.3.5 immediately follows from Prop.A.3. \square

3.3 The space of initial conditions

In this section, we give an algorithm to construct the space of initial conditions for a differential equation having the Painlevé property. For a polynomial system, a manifold $\mathcal{M}(z)$ is called the space of initial conditions if any solutions of the system give global holomorphic sections of the fiber bundle $\mathcal{P} = \{(x, z) \mid x \in \mathcal{M}(z), z \in \mathbb{C}\}$ over \mathbb{C} . Okamoto [10] constructed the spaces of initial conditions for the first to the sixth Painlevé equations by blow-ups of a Hirzebruch surface eight times and by removing a certain divisor called vertical leaves. In Chiba [3], the spaces of initial conditions for the first, second and fourth Painlevé equations, respectively, are obtained only by one, two and three times blow-ups with the aid of the weighted projective spaces. The purpose of this section is to extend this result. If a given equation having the Painlevé property has n -types Laurent series solutions (i.e. the equation $-p_i c_i = f_i^A(c_1, \dots, c_m)$ to determine the leading coefficients has n roots, and the corresponding series solutions are convergent Laurent series), then the space of initial conditions is obtained by n times weighted blow-up of the weighted projective space.

Suppose that the system (3.1) satisfying (A1) to (A3) is given. Determine the leading coefficients (c_1, \dots, c_m) of the formal series solution (2.7) by solving $-p_i c_i = f_i^A(c_1, \dots, c_m)$. For each (c_1, \dots, c_m) , we suppose that (2.7) is a convergent Laurent series (without $\log(z - z_0)$), so that the conditions (i) to (iii) of Prop.3.5 are satisfied. In particular, $\lambda_2, \dots, \lambda_k$ satisfy $\operatorname{Re}(\lambda_i) \geq 0$ and $\lambda_{k+1}, \dots, \lambda_m$ are positive integers. Then, the fixed point (3.5) is a singularity of the foliation defined by integral curves; any Laurent series solutions pass through this point. For each fixed point, we will perform the resolution of singularities. The procedure is divided into five steps as follows;

Step 1. Due to Prop.2.7, $(c_1, \dots, c_m) \neq (0, \dots, 0)$. Assume $c_j \neq 0$. Move to the inhomogeneous coordinates on $\mathbb{C}^{m+1}/\mathbb{Z}_{p_j}$ by (2.24) to obtain (3.3);

$$\begin{cases} \frac{dX_i}{dt} = p_i X_i - p_j \frac{f_i + \varepsilon G_i}{f_j + \varepsilon G_j} & (i = 1, \dots, m; i \neq j), \\ \frac{dZ}{dt} = rZ - \frac{p_j \varepsilon}{f_j + \varepsilon G_j}, \\ \frac{d\varepsilon}{dt} = s\varepsilon. \end{cases} \quad (3.12)$$

Our procedure below is independent of how to rewrite the system (3.2) to an au-

onomous vector field. For example, it may be more convenient to use the polynomial system (3.4) instead of (3.3) when calculating the normal form at Step 3.

This vector field has a fixed point (singularity) (3.5). In order for the point to be the origin, put $\hat{X}_i = X_i - c_i c_j^{-p_i/p_j}$, which results in

$$\begin{cases} \frac{d\hat{\mathbf{X}}}{dt} = \tilde{J}\hat{\mathbf{X}} + \mathbf{F}(\hat{\mathbf{X}}, Z, \varepsilon), & \hat{\mathbf{X}} = (\hat{X}_1, \dots, \hat{X}_{j-1}, \hat{X}_{j+1}, \dots, \hat{X}_m), \\ \frac{dZ}{dt} = rZ + \varepsilon F_{m+1}(\hat{\mathbf{X}}, Z, \varepsilon), \\ \frac{d\varepsilon}{dt} = s\varepsilon, \end{cases} \quad (3.13)$$

where \mathbf{F} is a nonlinearity and $F_{m+1} = -p_j/(f_j + \varepsilon G_j)$. The matrix \tilde{J} is a submatrix of J , whose eigenvalues are nontrivial K-exponents $\lambda_2, \dots, \lambda_m$ (see Eq.(3.6)).

Step 2. If the Jacobi matrix \tilde{J} has eigenvalues $\lambda_2, \dots, \lambda_k$ having nonpositive real parts, transform (3.13) as

$$\begin{cases} \frac{d\mathbf{X}_s}{dt} = J_s \mathbf{X}_s + \mathbf{F}_s(\mathbf{X}_s, \mathbf{X}_u, Z, \varepsilon), & \mathbf{X}_s \in \mathbb{C}^{k-1}, \\ \frac{d\mathbf{X}_u}{dt} = J_u \mathbf{X}_u + \mathbf{F}_u(\mathbf{X}_s, \mathbf{X}_u, Z, \varepsilon), & \mathbf{X}_u \in \mathbb{C}^{m-k}, \\ \frac{dZ}{dt} = rZ + \varepsilon F_{m+1}(\mathbf{X}_s, \mathbf{X}_u, Z, \varepsilon), \\ \frac{d\varepsilon}{dt} = s\varepsilon, \end{cases} \quad (3.14)$$

by a linear transformation of $(\hat{X}_1, \dots, \hat{X}_m)$ (we need not change Z and ε), where real parts of eigenvalues of J_u and J_s are positive and nonpositive, respectively. We suppose that J_u is of the Jordan normal form. Because of Prop.3.5, it is semi-simple; $J_u = \text{diag}(\lambda_{k+1}, \dots, \lambda_m)$. Due to the unstable manifold theorem, the unstable manifold is expressed as a convergent power series of the form

$$\mathbf{X}_s = \varphi(\mathbf{X}_u) = \sum_{|n|=2}^{\infty} \mathbf{b}_n \mathbf{X}_u^n, \quad (3.15)$$

where n denotes a multi-index as usual. The coefficient vectors \mathbf{b}_n can be obtained by substituting it into Eq.(3.14). The system on the unstable manifold is given by

$$\begin{cases} \frac{d\mathbf{X}_u}{dt} = J_u \mathbf{X}_u + \mathbf{F}_u(\varphi(\mathbf{X}_u), \mathbf{X}_u, Z, \varepsilon), \\ \frac{dZ}{dt} = rZ + \varepsilon F_{m+1}(\varphi(\mathbf{X}_u), \mathbf{X}_u, Z, \varepsilon), \\ \frac{d\varepsilon}{dt} = s\varepsilon, \end{cases} \quad (3.16)$$

Step 3. Calculate the normal form of the first equation of (3.16) up to degree N to be determined;

Due to the normal form theory, there exists a polynomial transformation (near identity transformation)

$$\mathbf{X}_u \mapsto \mathbf{Y} = h_1(\mathbf{X}_u, Z, \varepsilon), \quad h_1(0) = 0, Dh_1(0) = \text{id}, \quad (3.17)$$

of degree N , which can be exactly calculated for a finite N , such that the first equation of Eq.(3.16) takes the form (see Prop.A.1)

$$\begin{cases} \frac{d\mathbf{Y}}{dt} = J_u \mathbf{Y} + \mathbf{G}_1(\mathbf{Y}, Z, \varepsilon) + \mathbf{G}_2(\mathbf{Y}, Z, \varepsilon), & \mathbf{Y} \in \mathbb{C}^{m-k}, \\ \frac{dZ}{dt} = rZ + \varepsilon F_{m+1}, \\ \frac{d\varepsilon}{dt} = s\varepsilon, \end{cases} \quad (3.18)$$

where \mathbf{G}_1 consists of resonance terms up to degree N , and $\mathbf{G}_2 \sim O(\|(\mathbf{Y}, Z, \varepsilon)\|^{N+1})$. If we assume the condition (iii) of Prop.3.5, $\mathbf{G}_1 = 0$. Nevertheless, we keep the term \mathbf{G}_1 to observe that what happen when a given system (3.1) does not have the Painlevé property. How to choose N will be explained in Sec.4.2.

Step 4. Weighted blow-up;

Now the origin of the $m-k+2$ dimensional system (3.18) on the unstable manifold is a singularity; Laurent series solutions under consideration lie on the unstable manifold and they approach to the origin as $z \rightarrow z_0$. In order to resolve the singularity, we introduce the weighted blow-up with the weight $(\lambda_{k+1}, \dots, \lambda_m, r, s)$. Roughly speaking, the weighted blow-up is a birational transformation $\pi : B \rightarrow \mathbb{C}^{m-k+2}$ whose exceptional divisor $\pi^{-1}(0)$ is the weighted projective space $\mathbb{C}P^{m-k+1}(\lambda_{k+1}, \dots, \lambda_m, r, s)$, and B is a line bundle over $\mathbb{C}P^{m-k+1}(\lambda_{k+1}, \dots, \lambda_m, r, s)$. Denote $\mathbf{Y} = (Y_{k+1}, \dots, Y_m)$. One of the local coordinates $(u_{k+1}, \dots, u_m, \zeta, w)$ of B is defined by

$$\begin{pmatrix} Y_{k+1} \\ \vdots \\ Y_m \\ Z \\ \varepsilon \end{pmatrix} = \begin{pmatrix} u_{k+1} w^{\lambda_{k+1}} \\ \vdots \\ u_m w^{\lambda_m} \\ \zeta w^r \\ w^s \end{pmatrix}. \quad (3.19)$$

Hence, w denotes a coordinate on a fiber and $(u_{k+1}, \dots, u_m, \zeta)$ is the inhomogeneous coordinates of the chart $\mathbb{C}^{m-k+1}/\mathbb{Z}_s$ of $\mathbb{C}P^{m-k+1}(\lambda_{k+1}, \dots, \lambda_m, r, s)$. In particular, the set $\{w = 0\} \subset \mathbb{C}P^{m-k+1}(\lambda_{k+1}, \dots, \lambda_m, r, s)$ is attached at infinity of the original chart $\mathbb{C}^{m+1} = \{(x_1, \dots, x_m, z)\}$.

The coordinate transformation between the original chart and the new coordinates $(\mathbf{X}_s, u_{k+1}, \dots, u_m, \zeta, w)$ is given by

$$\begin{cases} x_i = (c_i c_j^{-p_i/p_j} + h_2(\mathbf{X}_s, u_{k+1}, \dots, u_m, \zeta, w)) w^{-p_i}, & (i \neq j) \\ x_j = w^{-p_j}, \\ z = \zeta, \end{cases} \quad (3.20)$$

where h_2 is a polynomial with $h_2(0) = 0$ that is obtained by a finite step.

Due to the orbifold structure of the exceptional divisor $\mathbb{C}P^{m-k+1}(\lambda_{k+1}, \dots, s)$, the \mathbb{Z}_s action

$$(u_{k+1}, \dots, u_m, \zeta, w) \mapsto (\omega^{\lambda_{k+1}} u_{k+1}, \dots, \omega^{\lambda_m} u_m, \omega^r \zeta, \omega^{-1} w), \quad \omega := e^{2\pi i/s},$$

acts on the space $\{(u_{k+1}, \dots, u_m, \zeta, w)\}$. This is compatible with the \mathbb{Z}_s action (2.19) of the original chart; the one action induces the other through the transformation (3.20).

Put $J_u = \text{diag}(\lambda_{k+1}, \dots, \lambda_m)$ and $\mathbf{G}_i = (G_{i,k+1}, \dots, G_{i,m})$. By the blow-up (3.19), Eq.(3.18) is transformed into the system

$$\begin{cases} \frac{du_i}{dt} = w^{-\lambda_i}(G_{1,n} + G_{2,n}), \\ \frac{d\zeta}{dt} = wF_{m+1}, \\ \frac{dw}{dt} = w. \end{cases}$$

Using $\zeta = z$ and deleting t , we obtain the system

$$\begin{cases} \frac{du_i}{dz} = w^{-1-\lambda_i}(G_{1,i} + G_{2,i})/F_{m+1}, & i = k+1, \dots, m, \\ \frac{dw}{dz} = 1/F_{m+1}. \end{cases} \quad (3.21)$$

Since $1/F_{m+1} = -(f_j + \varepsilon G_j)/p_j$ is holomorphic in u_{k+1}, \dots, u_m, w and z , a singularity of the right hand side may arise only from the factor $w^{-1-\lambda_i}$.

Proposition 3.7. If N is sufficiently large, the function $w^{-1-\lambda_i}G_{2,i}$ is holomorphic in $u_{k+1}, \dots, u_m, w, z$, while $w^{-1-\lambda_i}G_{1,i}$ is of the form

$$w^{-1-\lambda_i}G_{1,i} = w^{-1} \times (\text{polynomial of } u_{k+1}, \dots, u_m, z).$$

If the conditions of Prop.3.5 are satisfied, $G_{1,n} = 0$ and the right hand side of Eq.(3.21) is holomorphic in $u_{k+1}, \dots, u_m, w, z$. Further, $1/F_{m+1} \neq 0$ when $w = 0$. Hence, there are no singularities of the foliation on the exceptional divisor $\{w = 0\}$.

As a result, the singularity of the foliation at the point (3.5) is resolved; $m-k+1$ -parameter family of integral curves that lie on the unstable manifold, all of which pass through the fixed point (3.5) in $(X_1, \dots, X_m, Z, \varepsilon)$ coordinates, intersect with the $m-k+1$ -dimensional exceptional divisor $\{w = 0\}$ at different points. Further, if all K-exponents other than -1 are positive integers, then the right hand of (3.21) is polynomial because a transcendental function may arise only from the expression of the unstable manifold $\mathbf{X}_s = \varphi(\mathbf{X}_u)$.

Proof. Since

$$\mathbf{G}_2(\mathbf{Y}, Z, \varepsilon) = \mathbf{G}_2(u_{k+1}w^{\lambda_{k+1}}, \dots, u_mw^{\lambda_m}, \zeta w^r, w^s)$$

and $\mathbf{G}_2 \sim O(\|(\mathbf{Y}, Z, \varepsilon)\|^{N+1})$, $w^{-1-\lambda_i} G_{2,i}$ is holomorphic in w if N is sufficiently large. On the other hand, since $G_{1,i}(\mathbf{Y}, Z, \varepsilon)$ consists of resonance terms, a monomial $\alpha := Y_{k+1}^{n_{k+1}} \cdots Y_m^{n_m} Z^{n_{m+1}} \varepsilon^{n_{m+2}}$ included in $G_{1,i}(\mathbf{Y}, Z, \varepsilon)$ satisfies

$$\lambda_{k+1} n_{k+1} + \cdots + \lambda_m n_m + r n_{m+1} + s n_{m+2} = \lambda_i.$$

Hence, $w^{-1-\lambda_i} \alpha$ becomes of order $1/w$ after the blow-up.

Let us confirm the last statement. When $w = 0$, we have $Y_i = Z = \varepsilon = 0$. This implies $X_i = c_i c_j^{-p_i/p_j}$ when $w = 0$. Recall that f_j in Eq.(3.3) implies $f_j = f_j(X_1, \dots, 1, \dots, X_m, Z)$. Therefore, we obtain

$$\begin{aligned} 1/F_{m+1}|_{w=0} &= -\frac{1}{p_j} (f_j + \varepsilon G_j)|_{w=0} \\ &= -\frac{1}{p_j} f_j^A(c_1 c_j^{-p_1/p_j}, \dots, 1, \dots, c_m c_j^{-p_m/p_j}) \\ &= -\frac{1}{p_j} c_j^{-(1+p_j)/p_j} f_j^A(c_1, \dots, c_m) = c_j^{-1/p_j} \neq 0. \quad \square \end{aligned}$$

Step 5. Divide (3.20) and (3.21) by the \mathbb{Z}_{p_j} -action;

If $p_j \neq 1$, (3.20) is not a one-to-one transformation. Recall that the group \mathbb{Z}_{p_j}

$$(X_i, Z, \varepsilon) \mapsto (e^{2\pi i \cdot p_i/p_j} X_i, e^{2\pi i \cdot r/p_j} Z, e^{2\pi i \cdot s/p_j} \varepsilon), \quad (i \neq j)$$

acts on the inhomogeneous coordinates on the lift of the chart $\mathbb{C}^{m+1}/\mathbb{Z}_{p_j}$ and Eq.(3.3) is invariant under the action due to the orbifold structure. This action induces a \mathbb{Z}_{p_j} action on the $(\mathbf{X}_s, u_{k+1}, \dots, w)$ -coordinates and Eq.(3.21) is invariant under the action;

$$\mathbb{Z}_{p_j} \curvearrowright \mathbb{C}_1^m := \{(\mathbf{X}_s, u_{k+1}, \dots, u_m, w)\}.$$

Obviously, the right hand sides of the transformation (2.24) are invariant under the \mathbb{Z}_{p_j} action. This shows that the right hand sides of the transformation (3.20) are rational invariants of the \mathbb{Z}_{p_j} action. Thus, if we divide \mathbb{C}_1^m by the action, (3.20) becomes a one-to-one rational transformation, which can be explicitly given by rewriting the right hand sides of (3.20) in terms of polynomial invariants of the action $\mathbb{Z}_{p_j} \curvearrowright \mathbb{C}_1^{m+1}$, one of which should be $W := w^{p_j}$. The original chart $M_0 := \{(x_1, \dots, x_m)\} \simeq \mathbb{C}^m$ and the quotient space $M_1 := \mathbb{C}_1^m/\mathbb{Z}_{p_j}$, which is a nonsingular algebraic variety, are glued by the one-to-one rational transformation (3.20) to give a nonsingular algebraic variety M_{01} . Eq.(3.1) together with (3.21) gives a holomorphic equation on M_{01} without singularities of the foliation.

Suppose that a given system with the Painlevé property (in the sense that any solutions are meromorphic) has n -types Laurent series solutions, all of whose leading terms are of the form $x_i \sim c_i(z - z_0)^{-p_i}$; that is, there are n roots of the equation $-p_i c_i = f_i^A(c_1, \dots, c_m)$. We perform Step 1 to Step 5 for all Laurent series to obtain the manifold $M_i \simeq \mathbb{C}^m/\mathbb{Z}_{p_j}$ and a holomorphic differential equation on it as in Step 5. Then, an algebraic variety $\mathcal{M}(z) := M_0 \cup M_1 \cup \dots \cup M_n$ parameterized by

$z \in \mathbb{C}$ gives the space of initial conditions for (3.1). Each solution defines a global holomorphic section of the fiber bundle $\{(x, z) \mid x \in \mathcal{M}(z), z \in \mathbb{C}\}$ and there are no singularities of the foliation on the bundle. See Chiba [3] for the detailed calculation for the first, second and fourth Painlevé equations, and Section 4 for the higher order first Painlevé equation.

4 The first Painlevé hierarchy

Define the operator \mathcal{L}_m by

$$\frac{d}{dz}\mathcal{L}_{m+1}[x] = \left(\frac{d^3}{dz^3} - 8x\frac{d}{dx} - 4\frac{dx}{dz} \right) \mathcal{L}_m[x], \quad \mathcal{L}_0[x] = 1, \quad (4.1)$$

where $x = x(z)$ is a function of $z \in \mathbb{C}$. The $2m$ -th order first Painlevé equation (the first Painlevé hierarchy) is defined to be $\mathcal{L}_m[x] = -4z$. Indeed, it is easy to verify that there is a polynomial P_m such that the equation is expressed as

$$x^{(2m)} = P_m(x, x', \dots, x^{(2m-2)}) + z, \quad x^{(i)} := \frac{d^i x}{dz^i}. \quad (4.2)$$

For example, we obtain

$$\begin{aligned} x'' &= 6x^2 + z, \\ x'''' &= 20xx'' + 10(x')^2 - 40x^3 + z, \end{aligned}$$

for $m = 1, 2$, respectively. We rewrite (4.2) as the $2m$ -dimensional system

$$(P_1)_m \begin{cases} x'_1 = x_2 \\ \vdots \\ x'_{2m-1} = x_{2m} \\ x'_{2m} = P_m(x_1, \dots, x_{2m-1}) + z. \end{cases} \quad (4.3)$$

This system satisfies the assumptions (A1) to (A3) with $g_i = 0$ and

$$(p_1, p_2, \dots, p_{2m}, r, s) = (2, 3, \dots, 2m+1, 2m+2, 2m+3).$$

Indeed, it is easy to prove by induction that the function P_m satisfies

$$P_m(\lambda^2 x_1, \lambda^3 x_2, \dots, \lambda^{2m} x_{2m-1}) = \lambda^{2m+2} P_m(x_1, x_2, \dots, x_{2m-1}) \quad (4.4)$$

for any $\lambda \in \mathbb{C}$. Thus, the system (4.3) induces a rational ODE on the weighted projective space $\mathbb{C}P^{2m+1}(2, \dots, 2m+3)$. In Shimomura[12], it is proved that the first Painlevé hierarchy has the Painlevé property in the sense that any solutions are meromorphic functions. The leading coefficients of the Laurent series solutions are given by

$$c_j(k) = (-1)^{j+1} j! \cdot b_0, \quad b_0 := \frac{1}{2} k(k+1), \quad (4.5)$$

for $k = 1, \dots, m$ [11]. Hence, the system (4.3) has m Laurent series solutions of the form

$$x_i(z) = c_i(k)(z - z_0)^{-p_i} + \sum_{n=1}^{\infty} a_{i,n}(k)(z - z_0)^{-p_i+n}.$$

4.1 The K-exponents of the first Painlevé hierarchy

For each Laurent series solution with (4.5), the K-exponents are defined, which are given as follows.

Theorem 4.1. The K-exponents of the system (4.3) associated with the Laurent series solution with (4.5) are given by the following $2m$ integers;

$$\begin{aligned} \lambda = & \quad 2, \quad 4, \quad \dots, 2m - 2k, \quad (m - k) \\ & 2k + 3, 2k + 5, \dots, 2m + 1, \quad (m - k) \\ & 2m + 4, 2m + 6, \dots, 2m + 2k + 2, \quad (k) \\ & -1, \quad -3, \quad \dots, -(2k - 1), \quad (k) \end{aligned}$$

Thus, the Laurent series solution includes $2m - k + 1$ free parameters (including z_0). In particular, the Laurent series for the case $k = 1$ includes $2m$ free parameters that represents a general solution.

In order to prove the theorem, we need a Hamiltonian form of the system. By putting $x \mapsto \lambda^2 x$ and $z \mapsto \lambda^{-1} z$ with $\lambda^{-2m-3} = 4^m$, Eq.(4.2) is transformed to the equation $x^{(2m)} = P_m(x, \dots, x^{(2m-2)}) + 4^m z$ due to (4.4). Further, by putting

$$\begin{cases} u_j = 4^{1-j}(x_{2j-1} - P_{j-1}(x_1, \dots, x_{2j-3})), \\ v_j = \frac{4^{1-j}}{2} \left(x_{2j} - \sum_{i=1}^{2j-3} \frac{\partial P_{j-1}}{\partial x_i}(x_1, \dots, x_{2j-3}) x_{i+1} \right), \end{cases} \quad (4.6)$$

(u_j, v_j) satisfies the system

$$\begin{cases} u'_j = 2v_j, \\ v'_j = 2u_{j+1} + 2u_1 u_j + 2w_j, \quad (j = 1, \dots, m), \end{cases} \quad (4.7)$$

where $u_{m+1} = 0$ and w_j is determined by the recursive relation

$$w_j = \frac{1}{2} \sum_{k=1}^j u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2} \sum_{k=1}^{j-1} v_k v_{j-k} + \delta_{jm} z.$$

The system (4.7) is introduced by Shimomura [12] to prove the Painlevé property. If we define the weighted degree of x_j by $\deg(x_j) = j + 1$, then Eqs.(4.4) and (4.6) provide $\deg(u_j) = 2j$ and $\deg(v_j) = 2j + 1$. This implies that the transformation

$$(x_1, \dots, x_{2m}, z) \mapsto (u_1, v_1, \dots, u_m, v_m, z)$$

defined by (4.6) is an automorphism on $\mathbb{C}P^{2m+1}(2, \dots, 2m+3)$. In particular, K-exponents of Eq.(4.3) are the same as those of (4.7) due to Thm.2.5 or Thm.3.4.

According to Takei [13], we further change coordinates by

$$u_j = (-1)^{j-1}Q_j, \quad v_j = 2(-1)^{m-j}(P_{m-j+1} + P_{m-j+2}Q_1 + \dots + P_mQ_{j-1}). \quad (4.8)$$

Then, (P_j, Q_j) satisfies the Hamiltonian system

$$\frac{dP_j}{dz} = -\frac{\partial H_m}{\partial Q_j}, \quad \frac{dQ_j}{dz} = \frac{\partial H_m}{\partial P_j}, \quad (j = 1, \dots, m), \quad (4.9)$$

where H_m is a polynomial Hamiltonian function. Since the weighted degrees are given by $\deg(P_j) = 2m+3-2j$ and $\deg(Q_j) = 2j$, the transformation

$$(u_1, v_1, \dots, u_m, v_m, z) \mapsto (P_1, Q_1, \dots, P_m, Q_m, z)$$

is an isomorphism from $\mathbb{C}P^{2m+1}(2, \dots, 2m+3)$ to

$$\mathbb{C}P^{2m+1}(2m+1, 2, \dots, 2m+3-2j, 2j, \dots, 3, 2m, 2m+2, 2m+3).$$

In particular, the K-exponents do not change. It is easy to verify the equality

$$H_m(\dots, \lambda^{2m+3-2j}P_j, \lambda^{2j}Q_j, \dots, \lambda^{2m+2}z) = \lambda^{2m+4}H_m(\dots, P_j, Q_j, \dots, z). \quad (4.10)$$

Thus Lemma 2.4 provides

Lemma 4.2. If λ is a K-exponent of the system (4.3), so is $\mu = 2m+3-\lambda$.

Because of this lemma, the existence of K-exponents in the fourth line in Thm.4.1 immediately follows from that of the third line.

Proof of Thm.4.1. The K-matrix of the system (4.3) is given by

$$K = \begin{pmatrix} 2 & 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & j+1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 2m & 1 \\ \frac{\partial P_m}{\partial x_1} & \frac{\partial P_m}{\partial x_2} & \dots & \frac{\partial P_m}{\partial x_j} & \frac{\partial P_m}{\partial x_{j+1}} & \dots & \frac{\partial P_m}{\partial x_{2m-1}} & 2m+1 \end{pmatrix},$$

where $\partial P_m/\partial x_j$ is estimated at the point

$$\mathbf{c}(k) = (c_1(k), \dots, c_{2m-1}(k)).$$

The eigen-equation is given by

$$\det(\lambda - K) = (\lambda - 2) \dots (\lambda - 2m - 1) - \sum_{i=1}^{2m-1} \frac{\partial P_m}{\partial x_i}(\mathbf{c}(k))(\lambda - 2) \dots (\lambda - i) = 0.$$

By the definition, P_m satisfies

$$\mathcal{L}_{m+1}[x] = -4 \left(x^{(2m)} - P_m(x, \dots, x^{(2m-2)}) \right), \quad \mathcal{L}_0[x] = 1.$$

Putting $x = \varphi_0 + \delta\varphi_1$ yields

$$\begin{aligned} \mathcal{L}_{m+1}[\varphi_0 + \delta\varphi_1] &= -4 \left(\varphi_0^{(2m)} - P_m(\varphi_0, \dots, \varphi_0^{(2m-2)}) \right) \\ &\quad - 4\delta \left(\varphi_1^{(2m)} - \sum_{i=1}^{2m-1} \frac{\partial P_m}{\partial x_i}(\varphi_0, \dots, \varphi_0^{(2m-2)}) \varphi_1^{(i-1)} \right) + O(\delta^2). \end{aligned}$$

If we put $\varphi_0(z) = b_0(z+1)^{-2}$ with $b_0 = k(k+1)/2$, then the first term in the right hand side vanishes because of Eq.(4.4). Since

$$\varphi_0^{(j)}(0) = (-1)^j(j+1)!b_0 = c_{j+1}(k),$$

we obtain

$$\mathcal{L}_{m+1}[\varphi_0 + \delta\varphi_1](0) = -4\delta \left(\varphi_1^{(2m)}(0) - \sum_{i=1}^{2m-1} \frac{\partial P_m}{\partial x_i}(\mathbf{c}(k)) \varphi_1^{(i-1)}(0) \right) + O(\delta^2).$$

Further, putting $\varphi_1(z) = b_0(z+1)^{\lambda-2}$ provides

$$\mathcal{L}_{m+1}[\varphi_0 + \delta\varphi_1](0) = -4\delta b_0 \cdot \det(\lambda - K) + O(\delta^2).$$

Therefore, if we set

$$\mathcal{L}_j[\varphi_0 + \delta\varphi_1](z) = f_j(z) + \delta g_j(z) + O(\delta^2), \quad f_0 = 1, g_0 = 0, \quad (4.11)$$

$\det(\lambda - K) = 0$ is equivalent to $g_{m+1}(0) = 0$.

Let us derive difference equations for f_j and g_j . Substituting (4.11) into the definition (4.1) of \mathcal{L}_{j+1} , we obtain

$$\begin{cases} f'_{j+1} = f'''_j - 8\varphi_0 f'_j - 4\varphi'_0 f_j, \\ g'_{j+1} = g'''_j - 8\varphi_0 g'_j - 4\varphi'_0 g_j - 8\varphi_1 f'_j - 4\varphi'_1 f_j. \end{cases} \quad (4.12)$$

If we set $f_j = A_j(z+1)^{-2j}$, the first equation yields

$$\begin{aligned} (2j+2)A_{j+1} &= 2j(2j+1)(2j+2)A_j - 16jb_0A_j - 8b_0A_j \\ &= 8(2j+1) \left(\frac{1}{2}j(j+1) - b_0 \right) A_j. \end{aligned}$$

Thus, we have

$$A_{j+1} = -4b_0 \prod_{l=1}^j \frac{4(2l+1)}{l+1} \left(\frac{1}{2}l(l+1) - b_0 \right), \quad b_0 = \frac{1}{2}k(k+1). \quad (4.13)$$

This is further rearranged as

$$A_j = (-1)^j 2^j \frac{(2j-1)!! \cdot (k+j)!}{j! \cdot (k-j)!}, \quad (j = 1, \dots, k), \quad (4.14)$$

and $A_j = 0$ for $j \geq k+1$.

Next, by putting $g_j = B_j(z+1)^{\lambda-2j}$, the second equation of (4.12) gives

$$B_{j+1} = \frac{(\lambda - (2j+2k+2))(\lambda - (2j+1))(\lambda - (2j-2k))}{\lambda - (2j+2)} B_j - \frac{4(\lambda - (4j+2))}{\lambda - (2j+2)} b_0 A_j.$$

Since $g_{m+1}(0) = 0$ if and only if $B_{m+1} = 0$, roots of $B_{m+1}(\lambda) = 0$ give the K-exponents. Since $A_j = 0$ for $j \geq k+1$, we obtain

$$\begin{aligned} B_{m+1} &= \frac{(\lambda - (2m+2k+2))(\lambda - (2m+1))(\lambda - (2m-2k))}{\lambda - (2m+2)} B_m \\ &= \frac{\prod_{j=k+1}^m (\lambda - (2j+2k+2)) \cdot (\lambda - (2j+1)) \cdot (\lambda - (2j-2k))}{(\lambda - (2m+2)) \cdot (\lambda - 2m) \cdots (\lambda - (2k+4))} B_{k+1}. \end{aligned} \quad (4.15)$$

Now we need two lemmas.

Lemma 4.3. For any $j \geq k$, $B_{j+1}(\lambda)$ is a polynomial in λ of degree $2j$.

Lemma 4.4. The equation $B_{k+1}(\lambda) = 0$ has k roots given by $\lambda = 2k+4, 2k+6, \dots, 4k+2$. In particular, there is a polynomial $C_{k+1}(\lambda)$ of degree k such that

$$B_{k+1} = (\lambda - (2k+4)) \cdots (\lambda - (4k+2)) C_{k+1}(\lambda). \quad (4.16)$$

Lemma 4.3 is trivial because $B_{j+1}(\lambda) = 0$ is equivalent to the eigen-equation $\det(\lambda - K) = 0$ for the $2j$ dimensional problem. If Lemma 4.4 is true, all factors in the denominator of (4.15) cancel and we obtain

$$B_{m+1} = \prod_{j=m+1-k}^m (\lambda - (2j+2k+2)) \prod_{j=k+1}^m (\lambda - (2j+1)) \prod_{j=k+1}^m (\lambda - (2j-2k)) \cdot C_{k+1}(\lambda).$$

In particular, we obtained the first three lines in Thm.4.1. This completes the proof because of Lemma 4.2.

Proof of Lemma 4.5. Put

$$P_j = \frac{(\lambda - (2j+2k+2))(\lambda - (2j+1))(\lambda - (2j-2k))}{\lambda - (2j+2)}, \quad Q_j = -\frac{2(\lambda - (4j+2))}{\lambda - (2j+2)}.$$

Then we have

$$\begin{aligned} B_{k+1} &= P_k B_k + Q_k k(k+1) A_k \\ &= P_k (P_{k-1} B_{k-1} + Q_{k-1} k(k+1) A_{k-1}) + Q_k k(k+1) A_k \\ &\vdots \\ &= P_k P_{k-1} \cdots P_1 B_1 + [Q_k A_k + P_k Q_{k-1} A_{k-1} + \cdots + P_k \cdots P_2 Q_1 A_1] k(k+1). \end{aligned}$$

Substituting P_j, Q_j and (4.14), we obtain

$$\frac{B_{k+1}(\lambda)}{k(k+1)} = \sum_{l=0}^k \frac{\lambda - (4k - 4l + 2)}{\lambda - (2k - 2l + 2)} (-1)^{k-l+1} 2^{k-l+1} \frac{(2k - 2l - 1)!! \cdot (2k - l)!}{(k - l)! \cdot l!} \times \\ \prod_{j=k-l+1}^k \frac{(\lambda - (2j + 2k + 2)) (\lambda - (2j + 1)) (\lambda - (2j - 2k))}{\lambda - (2j + 2)}.$$

Now we show that $\lambda = 4k + 2 - 2n$ is a root of $B_{k+1}(\lambda) = 0$ for $n = 0, 1, \dots, k - 1$. Substituting this value gives

$$\frac{B_{k+1}(\lambda)}{k(k+1)} = \sum_{l=0}^k \frac{2l - n}{k + l - n} (-1)^{k-l+1} 2^{k-l+1} \frac{(2k - 2l - 1)!! \cdot (2k - l)!}{(k - l)! \cdot l!} \times \\ \prod_{j=k-l+1}^k \frac{(-2) (j + n - k) (4k - 2n - 2j + 1) (3k - n - j + 1)}{2k - n - j}.$$

Since the factor $j + n - k$ becomes zero when $j = k - n$, which is possible only for $l = n + 1, \dots, k$,

$$\frac{B_{k+1}(\lambda)}{k(k+1)} = (-1)^k 2^{k+1} \sum_{l=0}^n \frac{2l - n}{k + l - n} \frac{(2k - 2l - 1)!! \cdot (2k - l)!}{(k - l)! \cdot l!} \times \\ \prod_{j=k-l+1}^k \frac{(j + n - k) (4k - 2n - 2j + 1) (3k - n - j + 1)}{2k - n - j}.$$

Define

$$F(l) := \frac{2l - n}{k + l - n} \frac{(2k - 2l - 1)!! \cdot (2k - l)!}{(k - l)! \cdot l!} \times \\ \prod_{j=k-l+1}^k \frac{(j + n - k) (4k - 2n - 2j + 1) (3k - n - j + 1)}{2k - n - j}.$$

Then, it is straightforward to prove that $F(l) = -F(n-l)$ and $F(n/2) = 0$ when n is an even number. This proves $B_{k+1}(\lambda) = 0$ for $\lambda = 4k + 2 - 2n$ with $n = 0, 1, \dots, k - 1$. \square

4.2 The space of initial conditions for the fourth order equation

The fourth order first Painlevé equation is given by

$$(P_1)_2 \begin{cases} x'_1 = x_2 \\ x'_2 = x_3 \\ x'_3 = x_4 \\ x'_4 = 20x_1x_3 + 10x_2^2 - 40x_1^3 + z. \end{cases} \quad (4.17)$$

In this section, we demonstrate how to construct the space of the initial conditions of this system. The system satisfies the assumptions (A1) to (A3) with the weight

$$(p_1, p_2, p_3, p_4, r, s) = (2, 3, 4, 5, 6, 7). \quad (4.18)$$

Thus, we give the system on the local chart $\mathbb{C}^5/\mathbb{Z}_7$ of the space $\mathbb{C}P^5(2, 3, \dots, 7)$. The system has the two families of Laurent series solutions

$$(I) \quad x_j(z) \sim (-1)^{j+1} j! (z - z_0)^{-p_j}, \quad (4.19)$$

$$(II) \quad x_j(z) \sim 3(-1)^{j+1} j! (z - z_0)^{-p_j}, \quad (4.20)$$

whose K-exponents are given by

$$(I) \quad \lambda = -1, 2, 5, 8,$$

$$(II) \quad \lambda = -1, -3, 8, 10,$$

respectively. In particular, the first one represents a general solution. We perform the resolution of singularity for each Laurent series.

(I) Let us consider the resolution of the Laurent series (I).

Step 1. The coordinate transformation between the original coordinates and the inhomogeneous coordinates on $\mathbb{C}^5/\mathbb{Z}_2$ are given by

$$x_1 = \varepsilon^{-2/7}, \quad x_2 = X_2 \varepsilon^{-3/7}, \quad x_3 = X_3 \varepsilon^{-4/7}, \quad x_4 = X_4 \varepsilon^{-5/7}, \quad z = Z \varepsilon^{-6/7}. \quad (4.21)$$

We express the system (4.17) in the new coordinates as a polynomial vector field of the form (3.4)

$$\begin{cases} dX_2/dt = 3X_2^2 - 2X_3, \\ dX_3/dt = 4X_3X_2 - 2X_4, \\ dX_4/dt = 5X_4X_2 - (40X_3 + 20X_2^2 - 80 + 2z), \\ dZ/dt = 6ZX_2 - 2\varepsilon, \\ d\varepsilon/dt = 7\varepsilon X_2. \end{cases} \quad (4.22)$$

This system has two fixed points

$$(X_2, X_3, X_4, Z, \varepsilon) = (2, 6, 24, 0, 0), \quad (2/\sqrt{3}, 2, 8/\sqrt{3}, 0, 0),$$

which correspond to the Laurent series (I) and (II), respectively. The eigenvalues of the Jacobi matrix at these fixed points are

$$\begin{aligned} \lambda &= 4, 10, 16, 12, 14, \\ \lambda &= -2\sqrt{3}, 16/\sqrt{3}, 20/\sqrt{3}, 12/\sqrt{3}, 14/\sqrt{3}, \end{aligned}$$

respectively. Since we have used the polynomial form (3.4) instead of (3.3), they differ from the K-exponents, r and s by a constant factor (multiplied by $1/2$ and

$\sqrt{3}/2$, they become 2, 5, 8, 6, 7 for the first one and $-3, 8, 10, 6, 7$ for the second one, respectively, which coincide with the K-exponents and r, s).

For the resolution of singularity of the first fixed point (I), put

$$(\hat{X}_2, \hat{X}_3, \hat{X}_4) = (X_2 - 2, X_3 - 6, X_4 - 24) \quad (4.23)$$

to obtain the system of the form (3.13).

Step 2. We introduce the linear transformation

$$\begin{cases} \hat{X}_2 = v_1, \\ \hat{X}_3 = 4v_1 + v_2, \\ \hat{X}_4 = 20v_1 + 3v_2 + v_3 + Z/2 - \varepsilon/2. \end{cases} \quad (4.24)$$

Then, we obtain the system of the form

$$\begin{cases} dv_1/dt = 4v_1 + F_1(v_1, v_2, v_3, Z, \varepsilon) \\ dv_2/dt = 10v_2 + F_2(v_1, v_2, v_3, Z, \varepsilon) \\ dv_3/dt = 16v_3 + F_3(v_1, v_2, v_3, Z, \varepsilon) \\ dZ/dt = 12Z + F_4(v_1, v_2, v_3, Z, \varepsilon) \\ d\varepsilon/dt = 14\varepsilon + F_5(v_1, v_2, v_3, Z, \varepsilon), \end{cases} \quad (4.25)$$

where F_1, \dots, F_5 are defined by

$$\begin{cases} F_1 = -2v_2 + 3v_1^2, \\ F_2 = -2v_3 - Z + \varepsilon + 4v_1^2 + 4v_1v_2, \\ F_3 = 8v_1^2 + 3v_1v_2 + 5v_1v_3 - v_1Z/2 + v_1\varepsilon, \\ F_4 = -2\varepsilon + 6v_1Z, \\ F_5 = 7v_1\varepsilon. \end{cases}$$

Note that we need not diagonalize the linear part; $-2v_2$ in F_1 , $-2v_3 - Z + \varepsilon$ in F_2 and -2ε in F_4 do not yield a singularity after the blow-up (see the next step).

Step 3. We calculate the normal form of the system (4.25) to delete several monomials included in F_1, F_2, F_3 . We define weighted degrees to be

$$\deg(v_1) = 2, \deg(v_2) = 5, \deg(v_3) = 8, \deg(Z) = 6, \deg(\varepsilon) = 7, \quad (4.26)$$

which are the same as the weights of the weighted blow-up done in Step 4. From the argument of Step 4 in Sec.3.3, it turns out that if a monomial α included in F_i satisfies $\deg(\alpha) < \deg(v_i) + 1$, then the monomial yields a factor $1/w^n$ for some $n \geq 1$ in the right hand side of the system after the blow-up. Hence, we have to remove such monomials by the normal form theory. Monomials which may yield the factor $1/w^n$ are v_1^2 in F_2 and $v_1^2, v_1v_2, v_1^3, v_1^4$ in F_3 (although v_1^3 and v_1^4 are not included in F_3 , they may appear after removing v_1^2 and v_1v_2). To remove them, we set

$$\begin{cases} y_1 = v_1, \\ y_2 = v_2 + a_1v_1^2, \\ y_3 = v_3 + a_2v_1^2 + a_3v_1v_2 + a_4v_1^3 + a_5v_1^4. \end{cases} \quad (4.27)$$

We can verify by a straightforward calculation that if we put

$$a_1 = 3, \quad a_2 = 1, \quad a_3 = -1/2, \quad a_4 = -1/2, \quad a_5 = 0,$$

then the system (4.25) is brought into

$$\begin{cases} dy_1/dt = 4y_1 - 2y_2 + 9y_1^2, \\ dy_2/dt = 10y_2 - 2y_3 - Z + \varepsilon - 9y_1y_2 + 44y_1^3, \\ dy_3/dt = 16y_3 + 6y_1y_3 + y_1\varepsilon/2 + y_2^2 - 7y_1^2y_2/2 \\ dZ/dt = 12Z - 2\varepsilon + 6y_1Z \\ d\varepsilon/dt = 14\varepsilon + 7y_1\varepsilon. \end{cases} \quad (4.28)$$

This system does not include a monomial α satisfying $\deg(\alpha) < \deg(v_i) + 1$ except for the diagonal part $(4y_1, 10y_2, 16y_3, 12Z, 14\varepsilon)$, which will be removed by the blow-up below.

Step 4. We employ the weighted blow-up by

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ Z \\ \varepsilon \end{pmatrix} = \begin{pmatrix} u_1w^2 \\ u_2w^5 \\ u_3w^8 \\ \zeta w^6 \\ w^7 \end{pmatrix}. \quad (4.29)$$

Then, we obtain the polynomial system as desired;

$$\begin{cases} \frac{du_1}{dz} = u_2w^2 - \frac{7}{2}u_1^2w, \\ \frac{du_2}{dz} = -22u_1^3 + 7u_1u_2w + u_3w^2 - \frac{1}{2}w + \frac{1}{2}z, \\ \frac{du_3}{dz} = u_1u_3w - \frac{1}{4}u_1 + \frac{7}{4}u_1^2u_2 - \frac{1}{2}u_2^2w, \\ \frac{dw}{dz} = -1 - \frac{1}{2}u_1w^2. \end{cases} \quad (4.30)$$

The coordinate transformation between the original coordinates and $(u_1, u_2, u_3, w, \zeta)$ is given by

$$\begin{cases} x_1 = w^{-2} \\ x_2 = (2 + u_1w^2)w^{-3}, \\ x_3 = (6 + 4u_1w^2 - 3u_1^2w^4 + u_2w^5)w^{-4} \\ x_4 = (24 + 20u_1w^2 - 10u_1^2w^4 + 3u_2w^5 - u_1^3w^6 \\ \quad - \frac{1}{2}w^7 + \frac{1}{2}u_1u_2w^7 + u_3w^8 + \frac{1}{2}w^6z)w^{-5}, \\ z = \zeta. \end{cases} \quad (4.31)$$

Step 5. Due to the orbifold structure of $\mathbb{CP}^5(2, 3, 4, 5, 6, 7)$, the \mathbb{Z}_2 action

$$(X_2, X_3, X_4, Z, \varepsilon) \mapsto (-X_2, X_3, -X_4, Z, -\varepsilon) \quad (4.32)$$

acts on the coordinates $(X_2, X_3, X_4, Z, \varepsilon)$. This induces the \mathbb{Z}_2 action on the (u_1, u_2, u_3, z, w) coordinates given by

$$(u_1, u_2, u_3, z, w) \mapsto \left(-u_1 - \frac{4}{w^2}, -u_2 - \frac{32u_1}{w^3} - \frac{64}{w^5}, -u_3 - \frac{z}{w^2} - \frac{4u_2}{w^3} + \frac{24u_1^2}{w^4} + \frac{32u_1}{w^6} + \frac{64}{w^8}, z, -w\right). \quad (4.33)$$

If we divide $\mathbb{C}_1^4 = \{(u_1, u_2, u_3, w)\}$ by this action, (4.31) becomes a one-to-one rational transformation which defines a smooth algebraic variety $\mathbb{C}_0^4 \cup \mathbb{C}_1^4/\mathbb{Z}_2$, where $\mathbb{C}_0^4 = \{(x_1, x_2, x_3, x_4)\}$ is the original chart.

(II) Next, we consider the resolution of the Laurent series (II).

Step 1. For the resolution of singularity of the second fixed point (II) of the system (4.22), put

$$(\hat{X}_2, \hat{X}_3, \hat{X}_4) = (X_2 - 2/\sqrt{3}, X_3 - 2, X_4 - 8/\sqrt{3}) \quad (4.34)$$

to obtain the system of the form (3.13).

Step 2. We introduce the linear transformation

$$\begin{cases} \hat{X}_2 = v_1, \\ \hat{X}_3 = 3\sqrt{3}v_1 + v_2 + \frac{3}{8}Z - \frac{5\sqrt{3}}{8}\varepsilon, \\ \hat{X}_4 = 25v_1 + \frac{5\sqrt{3}}{3}v_2 + v_3 + \frac{7\sqrt{3}}{8}Z - \frac{27}{8}\varepsilon. \end{cases} \quad (4.35)$$

Then, we obtain the system of the form

$$\begin{cases} dv_1/dt = -2\sqrt{3}v_1 - 2v_2 - \frac{3}{4}Z + \frac{5\sqrt{3}}{4}\varepsilon + F_1(v_1, v_2, v_3, Z, \varepsilon) \\ dv_2/dt = \frac{16}{\sqrt{3}}v_2 - 2v_3 + F_2(v_1, v_2, v_3, Z, \varepsilon) \\ dv_3/dt = \frac{20}{\sqrt{3}}v_3 + F_3(v_1, v_2, v_3, Z, \varepsilon) \\ dZ/dt = \frac{12}{\sqrt{3}}Z - 2\varepsilon + F_4(v_1, v_2, v_3, Z, \varepsilon) \\ d\varepsilon/dt = \frac{14}{\sqrt{3}}\varepsilon + F_5(v_1, v_2, v_3, Z, \varepsilon), \end{cases}$$

where F_1, \dots, F_5 are nonlinear terms. The unstable manifold is a $(v_2, v_3, Z, \varepsilon)$ -space. We denote the unstable manifold by

$$v_1 = \varphi(v_2, v_3, Z, \varepsilon), \quad (4.36)$$

with a convergent power series φ which does not include a constant term. The system on the unstable manifold is given by

$$\begin{cases} dv_2/dt = \frac{16}{\sqrt{3}}v_2 - 2v_3 + F_2(\varphi(v_2, v_3, Z, \varepsilon), v_2, v_3, Z, \varepsilon) \\ dv_3/dt = \frac{20}{\sqrt{3}}v_3 + F_3(\varphi(v_2, v_3, Z, \varepsilon), v_2, v_3, Z, \varepsilon) \\ dZ/dt = \frac{12}{\sqrt{3}}Z - 2\varepsilon + F_4(\varphi(v_2, v_3, Z, \varepsilon), v_2, v_3, Z, \varepsilon) \\ d\varepsilon/dt = \frac{14}{\sqrt{3}}\varepsilon + F_5(\varphi(v_2, v_3, Z, \varepsilon), v_2, v_3, Z, \varepsilon). \end{cases} \quad (4.37)$$

Step 3. We define weighted degrees by

$$\deg(v_2) = 8, \deg(v_3) = 10, \deg(Z) = 6, \deg(\varepsilon) = 7, \quad (4.38)$$

which are the same as the weights of the weighted blow-up done in Step 4. As before, if a monomial α included in F_i ($i = 2, 3$) satisfies $\deg(\alpha) < \deg(v_i) + 1$, then the monomial yields a factor $1/w^n$ in the right hand side of the system after the blow-up. Since F_i is nonlinear, $F_i(\varphi(v_2, v_3, Z, \varepsilon), v_2, v_3, Z, \varepsilon)$ does not include such monomials (the possible least degree among nonlinear monomials is $\deg(Z^2) = 12$, which is larger than $\deg(v_2) + 1$ and $\deg(v_3) + 1$). Hence, we need not calculate the normal form.

Step 4. We employ the weighted blow-up by

$$\begin{pmatrix} v_2 \\ v_3 \\ Z \\ \varepsilon \end{pmatrix} = \begin{pmatrix} u_2 w^8 \\ u_3 w^{10} \\ \zeta w^6 \\ w^7 \end{pmatrix}. \quad (4.39)$$

The coordinate transformation between the original coordinates and $(v_1, u_2, u_3, w, \zeta)$ is given by

$$\begin{cases} x_1 = w^{-2} \\ x_2 = \left(\frac{2\sqrt{3}}{3} + v_1 \right) w^{-3}, \\ x_3 = \left(2 + 3\sqrt{3}v_1 + u_2 w^8 - \frac{5\sqrt{3}}{8} w^7 + \frac{3}{8} z w^6 \right) w^{-4}, \\ x_4 = \left(\frac{8\sqrt{3}}{3} + 25v_1 + \frac{5\sqrt{3}}{3} u_2 w^8 + u_3 w^{10} + \frac{7\sqrt{3}}{8} z w^6 - \frac{27}{8} w^7 \right) w^{-5}, \\ z = \zeta. \end{cases} \quad (4.40)$$

The equations of v_1, u_2, u_3, w are

$$\begin{cases} \frac{dv_1}{dz} = \frac{\sqrt{3}v_1}{w} - \frac{3v_1^2}{2w} - \frac{5\sqrt{3}}{8} w^6 + u_2 w^7 + \frac{3}{8} z w^5, \\ \frac{du_2}{dz} = -\frac{3\sqrt{3}v_1^2}{2w^9} - \frac{15\sqrt{3}v_1}{16w^2} + \frac{2v_1 u_2}{w} + u_3 w + \frac{3u_1 z}{8w^3}, \\ \frac{du_3}{dz} = -\frac{15v_1^2}{2w^{11}} + \frac{21v_1}{16w^4} - \frac{5\sqrt{3}v_1 u_2}{6} + \frac{5v_1 u_3}{2w} - \frac{3\sqrt{3}v_1 z}{16w^5}, \\ \frac{dw}{dz} = -\frac{\sqrt{3}}{3} - \frac{1}{2} v_1. \end{cases} \quad (4.41)$$

Although the right hand sides are not holomorphic at $w = 0$, they are holomorphic on the unstable manifold $v_1 = \varphi(u_2 w^8, u_3 w^{10}, z w^6, w^7) \sim O(w^6)$. Any integral curves of the vector field outside the unstable manifold approach to the other fixed point (I).

Step 5. The \mathbb{Z}_2 action (4.32) induces the \mathbb{Z}_2 action on the (v_1, u_2, u_3, z, w) coordinates given by

$$(v_1, u_2, u_3, z, w) \mapsto \left(-v_1 - \frac{4\sqrt{3}}{3}, u_2 - \frac{5\sqrt{3}}{4w} + \frac{6\sqrt{3}v_1}{w^8} + \frac{12}{w^8}, \right. \quad (4.42)$$

$$\left. -u_3 - \frac{10\sqrt{3}u_2}{3w^2} + \frac{25}{4w^3} - \frac{7\sqrt{3}z}{4w^4} + \frac{8\sqrt{3}}{w^{10}} - \frac{30v_1}{w^{10}}, z, -w\right). \quad (4.43)$$

If we divide $\mathbb{C}_2^4 = \{(v_1, u_2, u_3, w)\}$ by this action, (4.40) becomes a one-to-one rational transformation which defines a smooth algebraic variety $\mathbb{C}_0^4 \cup \mathbb{C}_2^4 / \mathbb{Z}_2$. Therefore, $\mathcal{M}(z) = \mathbb{C}_0^4 \cup \mathbb{C}_1^4 / \mathbb{Z}_2 \cup \mathbb{C}_2^4 / \mathbb{Z}_2$ defined by the transformations (4.31) and (4.40) gives the space of initial conditions for the system (4.14).

Acknowledgements.

The author would like to thank Professor Yasuhiko Yamada for useful comments. This work was supported by Grant-in-Aid for Young Scientists (B), No.25800081 from MEXT Japan.

A Normal form theory

In this Appendix, we give a brief review of the normal form theory. See [5] for more detail.

Let us consider a holomorphic vector field

$$\frac{dx}{dt} = Ax + f(x), \quad x \in \mathbb{C}^m, \quad (A.1)$$

defined near the origin, where A is an $m \times m$ matrix and $f \sim O(\|x\|^2)$ denotes the nonlinearity. We assume that $A = \text{diag}(\lambda_1, \dots, \lambda_m)$ is a diagonal matrix. If there exist j and non-negative integers (n_1, \dots, n_m) such that $n_1 + \dots + n_m \geq 2$ and

$$\lambda_1 n_1 + \dots + \lambda_m n_m = \lambda_j, \quad (A.2)$$

then, the monomial vector field $x_1^{n_1} \dots x_m^{n_m} e_j$ is called the resonance term. A normal form of (A.1) up to the order N is given as follows.

Proposition A.1. For any integer $N \geq 2$, there exists a polynomial transformation $x \mapsto y$ of degree N such that (A.1) is transformed into the system

$$\frac{dy}{dt} = Ay + g_1(y) + g_2(y), \quad (A.3)$$

where g_1 consists only of resonance terms up to degree N , and $g_2 \sim O(\|x\|^{N+1})$.

We need the following assumption for the convergence as $N \rightarrow \infty$.

(P) The convex hull of eigenvalues $\{\lambda_1, \dots, \lambda_m\}$ in \mathbb{C} does not include the origin.

In this case, the number of resonance terms is finite.

Proposition A.2. Under the assumption (P), there exists a local analytic transformation $x \mapsto y$ such that (A.1) is transformed into the system

$$\frac{dy}{dt} = Ay + g_1(y), \quad (\text{A.4})$$

where g_1 consists only of resonance terms.

(A.1) has a *formal* series solution of the form

$$x(t) = P(\alpha_1 e^{\lambda_1 t}, \dots, \alpha_m e^{\lambda_m t}), \quad (\text{A.5})$$

where P is a formal power series in the arguments, whose coefficients are polynomials in t . $\alpha_1, \dots, \alpha_m$ are arbitrary constants. The next proposition is well known in perturbation theory.

Proposition A.3. P is a *convergent* power series, whose coefficients are independent of t , if and only if

- (i) A is semi-simple, and
- (ii) (A.1) is linearized by a local analytic transformation.

In particular, the condition (ii) is satisfied if (P) is satisfied and $f(x)$ does not include resonance terms. There are examples that (P) is not satisfied while (A.1) can be linearized (Siegel's theorem). In the proof of Prop.3.5, the system (3.9) satisfies (P) because the eigenvalues have positive real parts.

B Proofs of Proposition 2.7 and 2.8

Consider the series solution

$$x_i(z) = c_i(z - z_0)^{-q_i} + a_{i,1}(z - z_0)^{-q_i+1} + \dots = c_i T^{-q_i}(1 + o(T)), \quad (\text{B.1})$$

where $T = z - z_0$. Without loss of generality, we suppose that

$$\frac{q_1}{p_1} = \dots = \frac{q_M}{p_M} > \frac{q_{M+1}}{p_{M+1}} \geq \dots \geq \frac{q_m}{p_m}. \quad (\text{B.2})$$

Substituting (B.1) into f_i^A , we have

$$\begin{aligned} & f_i^A(x_1(z), \dots, x_m(z)) \\ &= f_i^A(c_1 T^{-q_1}, \dots, c_m T^{-q_m}) \cdot (1 + o(T)) \\ &= f_i^A(c_1 T^{\frac{q_M}{p_M} p_1 - q_1} T^{-\frac{q_M}{p_M} p_1}, \dots, c_m T^{\frac{q_M}{p_M} p_m - q_m} T^{-\frac{q_M}{p_M} p_m}) \cdot (1 + o(T)) \\ &= T^{-\frac{q_M}{p_M}(1+p_i)} f_i^A(c_1 T^{\frac{q_M}{p_M} p_1 - q_1}, \dots, c_m T^{\frac{q_M}{p_M} p_m - q_m}) \cdot (1 + o(T)). \end{aligned}$$

(B.2) gives $q_M p_i / p_M - q_i \geq 0$ for any i and $q_M p_i / p_M - q_i = 0$ for $i = 1, \dots, M$. Thus we obtain

$$f_i^A(x_1(z), \dots, x_m(z)) = T^{-\frac{q_M}{p_M}(1+p_i)} f_i^A(c_1, \dots, c_M, 0, \dots, 0) \cdot (1 + o(T)).$$

Similarly, we can verify

$$\begin{aligned} f_i^N(x_1(z), \dots, x_m(z), z) &\sim o(T^{-\frac{q_M}{p_M}(1+p_i)}), \\ g_i(x_1(z), \dots, x_m(z), z) &\sim o(T^{-\frac{q_M}{p_M}(1+p_i)}), \end{aligned}$$

as $T \rightarrow 0$. Hence, the system (2.1) with (B.1) yields

$$-c_i q_i T^{-q_i-1}(1+o(T)) = T^{-\frac{q_M}{p_M}(1+p_i)} f_i^A(c_1, \dots, c_M, 0, \dots, 0) \cdot (1+o(T)). \quad (\text{B.3})$$

We will compare the orders of a pole in both sides; $q_i + 1$ and $q_M(1+p_i)/p_M$.

Proof of Prop.2.8. Assume that $q_i > p_i$ for some i . We can assume without loss of generality that $q_i > p_i$ for any $i = 1, \dots, m$. Then, (B.2) shows

$$(q_i + 1) - \frac{q_M}{p_M}(1+p_i) \leq (q_i + 1) - \frac{q_i}{p_i}(1+p_i) = 1 - \frac{q_i}{p_i} < 0. \quad (\text{B.4})$$

This proves

$$f_i^A(c_1, \dots, c_M, 0, \dots, 0) = 0$$

for $i = 1, \dots, m$. Then, the condition (S) gives $c_1 = \dots = c_M = 0$. We repeat this procedure by replacing $q_i - 1$ by q_i ($q_i - 1 := q_i$) for $i = 1, \dots, M$ and rearranging the order of q_1, q_2, \dots, q_m so that (B.2) holds for some M . If $q_i > p_i$, the inequality (B.4) again holds. If $q_i = p_i$ for some i , then

$$(q_i + 1) - \frac{q_M}{p_M}(1+p_i) < (q_i + 1) - \frac{q_i}{p_i}(1+p_i) = 0.$$

Since the inequality (B.4) holds for any i , we have $c_1 = \dots = c_M = 0$.

By repeating this procedure, at least q_1 decreases by 1 at each step. This algorithm stops when $q_i = p_i$ for any i and it completes the proof. \square

Proof of Prop.2.7. Suppose $0 \leq q_i < p_i$ and $c_i \neq 0$ for $i = 1, \dots, m$. For $i = 1, \dots, M$, we have

$$(q_i + 1) - \frac{q_M}{p_M}(1+p_i) = (q_i + 1) - \frac{q_i}{p_i}(1+p_i) = 1 - \frac{q_i}{p_i} > 0.$$

Thus, Eq.(B.3) gives $c_i q_i = 0$ for $i = 1, \dots, M$. By the assumption, we obtain $q_i = 0$ for $i = 1, \dots, M$. By repeating this procedure as before, we can prove that $q_i = 0$ for any i . \square

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